Varieties of groups and cellular covers

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Dedicated to my beloved parents.

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Contents

Chapter 1

Introduction

In recent years, there has been an increasing interest in the concept of cellular covers of groups. The notion of A-cellular objects in the category of groups was first introduced in [33]. Many initially significant results were later published in 2007 by E. D. Farjoun, R. Göbel and Y. Segev [7]. They mainly studied cellular covers of nilpotent groups and finite groups, and also some properties of groups inherited by their cellular covers. It was observed in this paper that cellular covers of nilpotent groups of class n are nilpotent of class n as well, in particular, cellular covers of abelian groups are abelian, see [7, pp. 62–63, Theorem 1.4]. This shows that the variety of nilpotent groups of class n is closed under taking cellular covers as well as the variety $\mathfrak{A}_p$ of abelian groups of prime exponent p. Hence, we obtain examples of countably many distinct varieties which are closed under taking cellular covers. Some more studies on cellular covers of particular groups and of groups with specific additional properties have been conducted and can be found in a considerable amount of literature, see e.g. [4], [5], [8], [11], [12], [13], and [34]. This area of research is still active with many interesting open questions.

In 2010, R. Göbel examined cellular covers of R-modules and of varieties of groups in [13], and established an example of countably many distinct varieties which are not closed under taking cellular covers by considering the Burnside variety $\mathfrak{B}_p$ of exponent p for primes $p > 10^{75}$:

Observation 1.1. *It is clear that the only abelian subvarieties of $\mathfrak{B}_p$ are the trivial variety $\{1\}$ and the variety $\mathfrak{A}_p$ of abelian groups of exponent p. But the variety*

$$\operatorname{cell}\mathfrak{B}_p := \langle H \mid H \textit{ is a cellular cover of some } G \in \mathfrak{B}_p\rangle$$

contains (for any prime $p > 10^{75}$) all abelian groups. Thus $\mathfrak{B}_p$ is not cellular closed.

However, the question of whether or not there exist uncountably many such varieties was left open, and it is natural to ask:

Are there $2^{\aleph_0}$ varieties of groups which are not closed under cellular covers?

This constitutes the main motivation of this thesis, where we will establish $2^{\aleph_0}$ distinct varieties of groups which are not cellular closed by applying the existence of a special Burnside group $\mathcal{B}$ of exponent p for any sufficiently large prime p and A. Yu. Ol'shanskii's remarkable result from 1970 (see [29]) quoted in the following

Theorem 1.2. *There exists exactly a continuum of distinct varieties of groups. Indeed, a continual set of varieties exists already among the varieties of length 5 solvable groups with exponent $e = 8p_1p_2$, where p_1 and p_2 are arbitrary relatively prime odd numbers.*

We will extend the $2^{\aleph_0}$ varieties of groups provided by A. Yu. Ol'shanskii by multiplying them by the Burnside variety $\mathfrak{B}_{p_3}$ for some sufficiently large prime p_3. That this results in varieties which are not cellular closed will be witnessed by the special Burnside group $\mathcal{B}$. Furthermore, that this construction results in $2^{\aleph_0}$ distinct varieties is guaranteed by the cancellation law of product varieties, see H. Neumann [27].

This thesis is divided into 7 chapters. All homomorphisms are written on the right hand side unless otherwise stated. In Chapter 2 we collect relevant basics about groups and varieties of groups that we need to apply in this thesis. The reader may skip this chapter if he/she is familiar with them. In Chapter 3 we deal with some special types of varieties, namely non-finitely based varieties, product varieties, and Burnside varieties and investigate their important properties. Next, all necessary fundamental knowledge about cellular covers of groups from [13] is collected in Chapter 4. Moreover, we also prove in this chapter the uniqueness of the A-cellular cover of G for any pair of groups A and G and we describe how this cellular cover of G can be explicitly constructed. In Chapter 5 we obtain the first new result, i.e., some explicit examples of Burnside varieties which are neither cellular closed nor finitely based. This requires Ol'shanskii's test groups for non-finitely based varieties and some knowledge about Schur multipliers. A more significant result, the existence of $2^{\aleph_0}$ distinct varieties of groups not closed under cellular covers, follows in Chapter 6, and the discussion of a consequence for order embedding into the lattice of all group varieties is included in this chapter. Finally, the existence of $2^{\aleph_0}$ distinct varieties of groups not closed under localizations, which acts as a dual case of our main result, is presented in Chapter 7.

Chapter 2

Preliminaries

Some elementary and necessary definitions and theorems from many different references are collected in this chapter for the convenience of the reader in order to make this thesis as self-contained as possible.

2.1 Basics about groups

If G is a group, and $g, h \in G$, then $[g, h] := g^{-1}h^{-1}gh$ and

$$G' := [G, G] := \langle [g, h] \mid g, h \in G \rangle$$

denotes the *derived* or *commutator subgroup* of G. More generally, we can define

$$[H, K] := \langle [h, k] \mid h \in H \text{ and } k \in K \rangle$$

for arbitrary subgroups H and K of G.

We recall a classical elementary result on derived subgroups, see e.g. Kurosh [24, p. 101].

Theorem 2.1.1. *The factor group G/N of a group G by a normal subgroup N is abelian if and only if $G' \subseteq N$.*

A group G is *solvable* if it has a subnormal series whose factors are abelian, i.e., there is a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_n = G$$

such that G_{i+1}/G_i is abelian for $i = 0, \ldots, n-1$. The shortest length of such a series is called the *solvable length* of G. For any group G the *derived* or *commutator series* of G is defined inductively:

$$G^{(0)} := G, \quad G^{(1)} := G' = [G, G] \quad \text{and} \quad G^{(i+1)} := [G^{(i)}, G^{(i)}]$$

for all $i \geq 1$. It is well-known that solvable groups can be approached also via their derived series. That is, A group G is solvable if and only if $G^{(n)} = 1$ for some $n \geq 0$, see [9, p. 195] for a reference.

Observe that subgroups and epimorphic images of solvable groups are again solvable, cf. [32, p. 121, Theorem 5.1.1]. In the following we collect some helpful criteria for checking solvable groups. We start with a classical result, cf. [32, p. 247, Theorem 8.5.3].

Theorem 2.1.2. (The Burnside p-q Theorem) *If p and q are primes, and m and n are non-negative integers, then any group of order $p^m q^n$ is solvable.*

For the following two famous results we refer to [32, p. 148] and [32, p. 403].

Theorem 2.1.3. (The Feit-Thompson Theorem) *Any finite group of odd order is solvable.*

Theorem 2.1.4. (Schreier Conjecture) *For any non-abelian finite simple group the group of outer automorphisms is solvable.*

In this thesis the *center* of a group G is denoted by

$$\mathfrak{z}G := \{g \in G \mid gh = hg, \forall h \in G\}.$$

A group G is called *nilpotent* if it has an *upper central series*

$$1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \ldots \leq Z_n(G) = G$$

where

$$Z_1(G) = \mathfrak{z}G \quad \text{and} \quad Z_i(G) \subseteq Z_{i+1}(G)$$

such that

$$Z_{i+1}(G)/Z_i(G) = \mathfrak{z}(G/Z_i(G)).$$

The smallest such n is called the *nilpotence class* of G. For any group G we define the following sequence of subgroups:

$$G^0 := G, \quad G^1 := G' = [G, G] \quad \text{and} \quad G^{i+1} := [G, G^i].$$

The chain of groups

$$G = G^0 \geq G^1 \geq G^2 \geq \ldots$$

is called the *lower central series* of G. As known, a group G is nilpotent if and only if $G^n = 1$ for some $n \geq 0$, see e.g. [9, p. 194, Theorem 8], and the smallest such n is identical with the nilpotence class of G.

We have the following characterization of finite nilpotent groups.

Theorem 2.1.5. *Let G be a finite group. Then the following properties are equivalent:*

(i) G is nilpotent.

(ii) G satisfies the normalizer condition*, i.e., every proper subgroup H is properly contained in its normalizer $N_G(H) := \{g \in G \mid g^{-1}Hg = H\}$.*

(iii) Every maximal proper subgroup of G is normal.

(iv) G is the cartesian product of its Sylow subgroups.

As a reference for this result we refer to [32, p. 130, Theorem 5.2.4]. Recall that if a finite group G is of order $|G| = p^n m$ with the greatest common divisor $(p, m) = 1$, then any p-subgroup of G of order p^n (i.e., any maximal p-subgroup) is called a *Sylow p-subgroup* of G (see e.g. [32, p. 39]).

We will see later in Theorem 5.3.2 that any subgroup of a free group is free, and a similar result holds also for abelian groups, see e.g. [32, p. 100, Theorem 4.2.3].

Theorem 2.1.6. *If G is a free abelian group with basis X and H is a subgroup of G, then H is free abelian with some basis Y and $|\,Y\,| \le |\,X\,|$.*

Suppose that $N \trianglelefteq G$ and there is a subgroup H of G such that $G = NH$ and $N \cap H = 1$. Then G is said to be the *internal semidirect product* of N and H and is denoted by $G := N \rtimes H$.

Conversely, suppose that there are two groups N and H given together with a homomorphism $\alpha : H \longrightarrow \operatorname{Aut} N$, where we denote $\alpha_h := h\alpha$ and will write the homomorphisms α_h on the left hand side (i.e., $\alpha_{h_1}\alpha_{h_2}(n) = \alpha_{h_1h_2}(n)$). The set of all pairs (n, h), $n \in N$ and $h \in H$, with the operation

$$(n_1, h_1)(n_2, h_2) := (n_1\alpha_{h_1}(n_2), h_1h_2),$$

is a group which is called the *external semidirect product* of N and H with respect to α and is denoted by $G := N \rtimes_\alpha H$. The sets $\{(n,1) \mid n \in N\}$ and $\{(1,h) \mid h \in H\}$ are subgroups of G, and the maps $n \mapsto (n,1)$ for $n \in N$ and $h \mapsto (1,h)$ for $h \in H$ give us that

$$N \cong \{(n,1) \mid n \in N\} \text{ and } H \cong \{(1,h) \mid h \in H\}.$$

Thus, we can identify N and H as canonical subgroups of G. Observe that $hnh^{-1} = \alpha_h(n)$. Hence, together with the conventional notation

$$n^h := h^{-1}nh,$$

the multiplication on G can be rewritten as

$$(n_1,h_1)(n_2,h_2) = (n_1 n_2^{h_1^{-1}}, h_1 h_2).$$

In addition, we obtain that

$$1_G = (1_N, 1_H), (n,h)^{-1} = (h^{-1}n^{-1}h, h^{-1}) = ((n^{-1})^h, h^{-1}) \text{ and } N \trianglelefteq G.$$

Therefore, $G = N \rtimes_\alpha H$ can be regarded as the internal semidirect product of N and H and we will simply call it the *semidirect product* of N and H and shortly denote it by $G = N \rtimes H$ (see e.g. [32, pp. 27–28]).

The following result is well-known and useful for constructing semidirect products, see [9, p. 829, Theorem 39].

Theorem 2.1.7. (Schur's Theorem) *Let G be a finite group, N a normal subgroup of G. If $|N|$ and $[G : N]$ are relatively prime, then N has a* complement *E in G, i.e., $E \subseteq G$, $E \cap N = 1$ and $NE = G$.*

We can construct a special product of two given groups based on a semidirect product: Let G and H be groups. The *unrestricted wreath product* of G

and H denoted by $W := G \operatorname{Wr} H$ or $W := G \wr H$ is the external semidirect product $K \rtimes_\alpha H$ where $K := \prod_{h\in H} G_h$ and $\alpha_h(g_{h'}) := g_{hh'}$ for all $h' \in H$, $g_{h'} \in G_{h'}$. The *restricted wreath product* of G and H denoted by $W := G \operatorname{wr} H$ is defined in the same way but in this case $K := \bigoplus_{h\in H} G_h$.

In order to formulate Maschke's theorem (which we will apply proving Lemma 6.1.4), we need the following definitions, see also [32, pp. 213–215] and [9, p. 413 and p. 840].

Definition 2.1.8. *(i) If G is a group and V is a vector space over a field F, then an F-*representation *of G, also known as a* linear representation *of G over F, is a homomorphism ρ from G to $GL(V)$, the automorphism group of V.*

Note that if V is of finite dimension n, then by fixing a basis of V we know that $GL(V) \cong GL(n, F)$, the group of invertible $n \times n$ matrices with entries from F.

(ii) A subspace U of V such that $U(g\rho) \subseteq U$ for all $g \in G$ (i.e., U is invariant under the group action) is called a subrepresentation *of V, and V is said to be* irreducible *if it has only two subrepresentations, namely 0 and V itself.*

(iii) An F-representation is completely reducible*, if V is a direct sum of irreducible subrepresentations.*

(iv) An F-representation is faithful*, if it is injective.*

We refer to [32, p. 216, Theorem 8.1.2] for the next

Theorem 2.1.9. (Maschke's Theorem) *Let G be a finite group, F a field whose characteristic does not divide the order of G and V a finite dimensional vector space over F. Then every F-representation of G is completely reducible.*

The following definition describes a useful construction tool applied in Theorem 4.1.9 (see [27, p. 33, Definition 18.11]).

Definition 2.1.10. *The group $G := *_{\lambda\in\Lambda}A_\lambda$ is called the* (internal) free product *of its subgroups A_λ, $\lambda \in \Lambda$, if*

(i) it is generated by these subgroups and

(ii) to any given homomorphisms $\alpha_\lambda : A_\lambda \longrightarrow B$ of the groups A_λ into a group B there exists a homomorphism $\alpha : G \longrightarrow B$ whose restriction to A_λ coincides with α_λ for each $\lambda \in \Lambda$.

Every family A_λ, $\lambda \in \Lambda$, of groups defines up to isomorphism a group G such that $G = *_{\lambda\in\Lambda}A_\lambda$ holds. This is called the *(external) free product* of the groups A_λ, $\lambda \in \Lambda$ (see H. Neumann for how G is constructed).

Let G be a group with a set of generators $\{g_i\}_{i\in I}$ and F a group freely generated by $X = \{x_i\}_{i\in I}$. An epimorphism from F to G is defined by $x_i \mapsto g_i$. If N is the kernel of this homomorphism, we obtain, by the first homomorphism theorem, that $G \cong F/N$, but for convenience we will identify G and F/N and we will say that $G = F/N$ is a *free presentation* of G.

We say that G is defined by a set of relations $\{r = 1 \mid r \in \mathcal{R}\}$, $\mathcal{R} \subseteq F$, if any relation among the generators g_i of G is a consequence of $\{r = 1 \mid r \in \mathcal{R}\}$. Each relation $r = 1, r \in \mathcal{R}$, is then called a *defining relation* and each $r \in \mathcal{R}$ is called a *relator* for G. Strictly speaking, a relation is a word with variables from X but in practice we usually abuse the notation slightly by making no difference between the elements g_i and the letters x_i. We also make no difference between the relation $r = 1$ and the relator r. We thus may write

$$G = \langle X \mid \mathcal{R} \rangle$$

and obtain equivalently that $G = F/N$, where $N := \mathcal{R}^F$ is the *normal closure* of the set $\mathcal{R}$ in F, the smallest normal subgroup of F containing $\mathcal{R}$.

A sequence of groups and homomorphisms

$$1 \longrightarrow G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \longrightarrow 1$$

is called a *short exact sequence*, if f is injective, g is surjective and $\operatorname{Im} f = \operatorname{Ker} g$. Furthermore, it is called a *central extension*, if in addition $\operatorname{Im} f \subseteq \mathfrak{z}G_2$. Since $N/[F,N] \subseteq \mathfrak{z}(F/[F,N])$, the following short exact sequence

$$1 \longrightarrow N/[F,N] \longrightarrow F/[F,N] \longrightarrow G \longrightarrow 1$$

is a central extension for any free presentation $G = F/N$.

The next lemma will be a key to prove one of our main results in Theorem 5.4.6, see [22, p. 46, Lemma 2.4.1] for a reference.

Lemma 2.1.11. *Let $G = F/N$ be a free presentation,*

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

be a central extension and $\alpha : G \longrightarrow C$ be a homomorphism. Then there is a homomorphism

$$\beta : F/[F,N] \longrightarrow B$$

which makes the following diagram commute.

$$\begin{tikzcd}
1 \arrow[r] & N/[F,N] \arrow[r] \arrow[d, "\gamma"] & F/[F,N] \arrow[r] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "\alpha"] & 1 \\
1 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 1
\end{tikzcd}$$

The map γ is the restriction of β to $N/[F,N]$.

2.2 Basics on varieties of groups

We will adopt the following notations and basic results about varieties of groups, mainly following [27] and [32].

If $\mathfrak{X}$ is a class of groups, then we consider the following closure operators:

$\mathrm{S}\mathfrak{X}$ is the class of subgroups of groups in $\mathfrak{X}$,

$\mathrm{Q}\mathfrak{X}$ is the class of quotient groups of groups in $\mathfrak{X}$,

$\mathrm{C}\mathfrak{X}$ is the class of cartesian products of groups in $\mathfrak{X}$,

$\mathrm{D}\mathfrak{X}$ is the class of finite cartesian products of groups in $\mathfrak{X}$.

A *factor* of a group A is any factor group H/K where $1 \leq K \trianglelefteq H \leq A$. It is proper unless $K = 1$ and $H = A$. Hence, $\mathrm{QS}\mathfrak{X}$ consists of all factors of groups in $\mathfrak{X}$.

Definition 2.2.1. *A* variety *of groups is a class $\mathfrak{X}$ of groups closed under the operators* S*,* Q *and* C*. By* $\operatorname{var}\mathfrak{X}$ *we denote the variety generated by a given set or class $\mathfrak{X}$ of groups. It is the smallest variety containing the given class $\mathfrak{X}$.*

By Birkhoff's theorem any variety $\mathfrak{X}$ can equivalently be represented by a set of laws W. Here W is a subset of a free group F of countable rank, say $F = F(X_\infty)$ is generated by a countable set $X_\infty = \{x_i \mid i \in \omega\}$ of variables. Every law $w \in W$ represents the equation $w = 1$. Thus $G \in \mathfrak{X}$ if and only if $w(g_1, ..., g_n) = 1$ for all $w = w(x_1, ..., x_n) \in W$ and $g_i \in G$.

There is a special kind of groups, which will play a role in Section 6.1, called monolithic defined as follows.

Definition 2.2.2. *A group G is* monolithic *if G has a unique minimal normal subgroup $\neq 1$, and the minimal normal subgroup is called the* monolith *of G.*

We will use one more special kind of groups.

Definition 2.2.3. *We say that a group G is* critical *if G is finite and does not belong to the variety generated by its proper factors.*

Now the following result holds for critical groups.

Theorem 2.2.4. *Every critical group is monolithic.*

For a reference we refer to [27, p. 147, Theorem 51.32].

Recall that a group is *locally finite* if its finitely generated subgroups are finite, [31, p. 60]. A variety is *locally finite* if all its group are locally finite, cf. [27, p. 19]. Critical groups are closely related to varieties of groups and essential for locally finite varieties, see [27, pp. 149–150, Remark 51.41 and Remark 51.43] for a proof.

Remark 2.2.5. *Any locally finite variety is generated by its critical groups, i.e., if $\mathfrak{V}$ is locally finite, then $\mathfrak{V} = \mathrm{var}\{H \in \mathfrak{V} \mid H \text{ is critical}\}$.*

Remark 2.2.6. *Any finite group belongs to the variety generated by its critical factors.*

Also recall that

$$C_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}$$

denotes the *centralizer* of a subgroup H of G, and note that clearly $C_G(H)$ is normal in G if H is normal in G. In preparation for Lemma 2.2.9 we refer to [27, p. 162, Definition 53.11] for the following

Definition 2.2.7. *Let $M \trianglelefteq A$ and $N \trianglelefteq B$. We say that M is* similar *to N and write $(M \trianglelefteq A) \sim (N \trianglelefteq B)$ if there are isomorphisms $\mu : M \longrightarrow N$ and $\alpha : A/C_A(M) \longrightarrow B/C_B(N)$ such that for all $m \in M$ and $a \in A/C_A(M)$*

$$(m^a)\mu = (m\mu)^{a\alpha}.$$

$$\begin{tikzcd}
M \arrow[r, "\mu", "\cong"'] & N \\
A \arrow[u, dotted] \arrow[d, dotted] & B \arrow[u, dotted] \arrow[d, dotted] \\
A/C_A(M) \arrow[r, "\alpha", "\cong"'] & B/C_B(M)
\end{tikzcd}$$

The element m^a (and similarly $(m\mu)^{a\alpha}$) is well-defined, i.e., it is the common value of all conjugates m^x, $x \in aC_A(M)$.

We have a simple but important remark about finite groups in a locally finite variety (see [27, p. 145, Remark 51.1]).

Remark 2.2.8. *If $\mathfrak{D}$ is a locally finite variety generated by a family D of finite groups and A is a finite group in $\mathfrak{D}$, then $A \in {\scriptstyle QSD}\, D$, that is, A is a factor of a finite cartesian product of groups in D.*

Let $\mathfrak{D}$ be a locally finite variety generated by a set D of finite groups such that D is closed under taking factors. For any finite group A in $\mathfrak{D}$, by Remark 2.2.8, we have that A is a factor of a finite cartesian product $\prod_{i=1}^{n} D_i$ of groups in D. But in general there may be many possibilities to present A this way. We will choose a *minimal presentation of A in $\mathfrak{D}$* as follows: Each presentation of A determines a finite sequence of integers which consists of the cardinals $|\, D_i\,|$ listed in non-increasing order. For convenience, we will fill up any such sequence by ones to obtain an infinite non-increasing sequence of integers. Ordered lexicographically, i.e., a sequence comes before another if its entry at the first place where they differ is the smaller one, the set of these sequences has a uniquely determined first element. A minimal presentation of

A is any presentation corresponding to this particular sequence and we may write A as

$$A = H/K, \text{ where } K \trianglelefteq H \subseteq P = \prod_{i=1}^{n} D_i.$$

We refer to [27, p. 165, Lemmas 53.25 and 53.26] for the next

Lemma 2.2.9. *If $\mathfrak{D}$ is a locally finite variety, $A \in \mathfrak{D}$ is a finite, monolithic group and $A = H/K, H \subseteq \prod_{i=1}^{n} D_i$ is a minimal presentation of A in $\mathfrak{D}$, then the following holds:*

(i) The groups D_i are critical.

(ii) The monolith M of A in A is similar to the monolith M_i of D_i in D_i for each i:

$$(M \trianglelefteq A) \sim (M_i \trianglelefteq D_i).$$

(iii) The group A/M is contained in $\operatorname{var}\{D_i/M_i \mid 1 \leq i \leq n\}$.

Chapter 3

Examples of group varieties

In this chapter, we will study some more special types of varieties of groups.

3.1 Non-finitely based varieties

We will use the following definition for verbal subgroups, see e.g. Neumann [27, p. 5, Definition 12.21].

Definition 3.1.1. *Let $F = F(X_\infty)$ be a group freely generated by $X_\infty = \{x_i \mid i \in \omega\}$. If $V \leq F$ is a set, and G is any group, then $VG := \{w(g_1, \dots, g_n) \mid g_i \in G, w = w(x_1, \dots, x_n) \in V\} \leq G$ is the set of values obtained by substituting elements of G into w. Then $V(G) := \langle VG \rangle$ is the* verbal subgroup *of G generated by VG, and $\mathfrak{V}$ denotes the variety of groups defined by V, which is the class $\{G \mid V(G) = 1\}$.*

We say that the word w is a *law* in a group G if $w(G) = 1$. Recall that a subgroup H of G is *fully invariant* if it is invariant under all endomorphisms of G. It is interesting to note that the verbal subgroup $V(G)$ of G is fully invariant in G (see [27, p. 5, Remark 12.33]), hence it is normal in G and we can consider the quotient of G by its verbal subgroup $V(G)$, which is an element of $\mathfrak{V}$.

We recall the following

Observation 3.1.2. *Let $V, W \leq F = F(X_\infty)$. Then the following are equivalent:*

(i) V and W generate the same variety, $\mathfrak{V} = \mathfrak{W}$.

(ii) The verbal subgroups $V(G)$ and $W(G)$ are the same for all groups G.

(iii) The verbal subgroups $V(F)$ and $W(F)$ coincide.

In the same context it is clear (using substitution) that the equation $w = 1$ for $w(x_1, \ldots, x_k) \in F$ is a consequence of the equations $v = 1, v(x_1, \ldots, x_n) \in V$ if and only if $w \in V(F)$.

Definition 3.1.3. *A variety $\mathfrak{V}$ is* finitely based *if there is a finite set of words $w_i, (i \leq k)$ such that $G \in \mathfrak{V}$ if and only if $w_i(G) = 1$ for all $i \leq k$. If $k = 1$, then $\mathfrak{V}$ is* singly based.

We will need the following well-known facts. First we show the trivial

Proposition 3.1.4. *A variety $\mathfrak{V}$ is finitely based if and only if it is singly based.*

Proof. It is enough to show that the variety $\mathfrak{V}$ based on $w_i(x_{i_1}, \ldots, x_{i_{j(i)}})$ $(i \leq k)$ is singly based. We may assume that the free variables $x_{1_1}, \ldots, x_{k_{j(k)}}$ are all distinct, and it is easy to see that $\mathfrak{V}$ is also defined by the single law $w(x_{1_1}, \ldots, x_{k_{j(k)}}) := \prod_{i \leq k} w_i(x_{i_1}, \ldots, x_{i_{j(i)}})$; just replace the suitable variables by 1 to recover each law w_i from w. Hence, the variety $\mathfrak{V}$ is singly based. $\square$

There always exist in any variety $\mathfrak{V}$ some groups that are free with respect to this variety, called $\mathfrak{V}$-*free* groups: Let $F = F(X)$ be the group freely generated by $X = \{x_i \mid i \in I\}$ and $V(F)$ the verbal subgroup of F, where V defines $\mathfrak{V}$. Then $F/V(F) \in \mathfrak{V}$ is called the *free group on X in* $\mathfrak{V}$ (and the isomorphic images of $F/V(F)$ are called the *relatively free groups of* $\mathfrak{V}$ or $\mathfrak{V}$-*free groups*). This group is generated by the set $\{x_iV(F) \mid i \in I\}$ which is called a *basis* of the $\mathfrak{V}$-free group $F/V(F)$. Hence, we can see that the relatively free group in the variety of all abelian groups is $F/[F,F]$, the free abelian group. Furthermore, $F = F(X)$ is the relatively free group in the variety of all groups, which is defined by an empty set of laws.

We elaborate the following lemma and corollary given in [31, p. 58, Theorem 6.6 and p. 59, Corollary 6.2].

Lemma 3.1.5. *Let $V = \{v_i(x_{i_1}, \ldots, x_{i_{j(i)}}) \mid i \in \omega\}$ be an infinite set of laws, and suppose that $w(x_1, \ldots, x_n)$ is a consequence of V. Then w is a consequence of a finite subset of V.*

Proof. We have $w \in V(F)$, by assumption on w. Then w is a (finite) product of words $v_i^{\pm 1}(f_{i_1}, \ldots, f_{i_{j(i)}}) \in V(F)$, say for $i \leq k$. Therefore, w is a consequence of $v_1, \ldots, v_k$. $\square$

Corollary 3.1.6. *Let $V = \{v_i(x_{i_1}, \ldots, x_{i_{j(i)}}) \mid i \in \omega\}$ be an infinite set of laws, and suppose that for each $n < \omega$ the law v_{n+1} is not a consequence of the predecessor laws $v_i, 1 \leq i \leq n$, then there is no finite set of laws $W \leq F = F(X_\infty)$ such that $W(F) = V(F)$, so the variety $\mathfrak{V}$ is not finitely based.*

Proof. Assume for contradiction that some finite set of laws W with $W(F) = V(F)$ exists. By Proposition 3.1.4 we may assume that $\mathfrak{W} = \mathfrak{V}$ is singly based by the law w. By Lemma 3.1.5, we see that w is a consequence of $\{v_1, \ldots, v_n\}$ for some $n < \omega$, hence $v_{n+1} \in V$ (as a consequence of w) is a consequence of its predecessors, which is a contradiction. $\square$

3.2 Product varieties

We begin this section by recalling the following definitions given in [27, p. 6].

Definition 3.2.1. *A set of words V is* closed *if it is a fully invariant subgroup of $F = F(X_\infty)$. In more detail: $V \neq \emptyset$ is closed if and only if*

(i) if $v \in V$, then $v^{-1} \in V$,

(ii) if $v_1, v_2 \in V$, then $v_1 v_2 \in V$,

(iii) if $v = v(x_1, \ldots, x_n) \in V$ is a word in n variables and $(f_1, \ldots, f_n) \in F^n$, then $v(f_1, \ldots, f_n) \in V$.

Equivalently, V is closed if and only if $V = V(F)$.

We can see at once that the intersection of any number of closed sets is closed. Then it is reasonable to consider

Definition 3.2.2. *The* closure *of a set V of words is the intersection of all closed sets containing V.*

For what follows we will regard every set of laws defining a variety as a closed set, since a set of words and its closure give rise to the same variety (see [27, p. 6, Remark 12.51 and Remark 12.52]).

Recall that a group G is an *extension* of a group N by a group Q if G has a normal subgroup $M \cong N$ with $G/M \cong Q$, see e.g. [32, p. 68], or [36, p. 154]. Following [27, p. 38] we can extend two given varieties as in the next

Definition 3.2.3. *If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties of groups, then the* product $\mathfrak{U}\mathfrak{V}$ *is the variety of all extensions of a group in $\mathfrak{U}$ by a group in $\mathfrak{V}$.*

Before dealing with product varieties defined in this way, we will first check that the product $\mathfrak{U}\mathfrak{V}$ is indeed a variety, i.e., we will check that it is closed under the operators $\textsc{s}$, $\textsc{q}$ and $\textsc{c}$ respectively: Let G be a group in $\mathfrak{U}\mathfrak{V}$. Then there is a normal subgroup A of G such that $A \in \mathfrak{U}$ and $G/A \in \mathfrak{V}$. If $H \subseteq G$, then $H \cap A$ is normal in H, and $H \cap A \subseteq A \in \mathfrak{U}$ with

$$H/(H \cap A) \cong HA/A \subseteq G/A \in \mathfrak{V}.$$

This shows that $\mathfrak{U}\mathfrak{V}$ is closed under taking subgroups. To show that $\mathfrak{U}\mathfrak{V}$ is closed under taking quotient groups, let $H \trianglelefteq G$. Then $AH \trianglelefteq G$, hence $AH/H \trianglelefteq G/H$. Moreover, $AH/H \cong A/(H \cap A) \in \mathfrak{U}$ since $H \cap A \trianglelefteq A \in \mathfrak{U}$, and

$$(G/H)/(AH/H) \cong G/AH \cong (G/A)/(AH/A) \in \mathfrak{V}$$

since $AH/A \trianglelefteq G/A \in \mathfrak{V}$. In order to show that $\mathfrak{U}\mathfrak{V}$ is closed under taking cartesian products, let $G_i \in \mathfrak{U}\mathfrak{V}, i \in \kappa$, where κ is an arbitrary cardinal. Then there is, for each i, a normal subgroup A_i of G_i such that $A_i \in \mathfrak{U}$ and $G_i/A_i \in \mathfrak{V}$. It follows that $\prod_{i \in \kappa} A_i \in \mathfrak{U}$, because $\mathfrak{U}$ is a variety, and this product is normal in $\prod_{i \in \kappa} G_i$. Next, define a homomorphism

$$f : \prod_{i \in \kappa} G_i \longrightarrow \prod_{i \in \kappa} (G_i/A_i) \ ((g_1, g_2, \ldots) \mapsto (g_1 A_1, g_2 A_2, \ldots)).$$

It is easy to see that f is surjective with $\operatorname{Ker} f = \prod_{i\in\kappa} A_i$. Therefore, we obtain that

$$\prod_{i\in\kappa} G_i / \prod_{i\in\kappa} A_i \cong \prod_{i\in\kappa} (G_i/A_i) \in \mathfrak{V},$$

because $G_i/A_i \in \mathfrak{V}$ for every i.

As the extensions of locally finite groups by locally finite groups are again locally finite, see e.g. [32, p. 429, Theorem 14.3.1], we immediately have the following

Corollary 3.2.4. *If $\mathfrak{U}$ and $\mathfrak{V}$ are locally finite varieties of groups, then also $\mathfrak{U}\mathfrak{V}$ is locally finite.*

For given varieties $\mathfrak{U}$ and $\mathfrak{V}$, it is helpful to know a set of laws of the product $\mathfrak{U}\mathfrak{V}$. Thus we will elaborate the proof of the following lemma given in [27, p. 38, Theorem 21.12].

Theorem 3.2.5. *If the closed sets U and V represent the laws of $\mathfrak{U}$ and $\mathfrak{V}$ respectively, then the verbal subgroup $U(V) = \langle w \mid w = u(v_1, \ldots, v_n), u = u(x_1, \ldots, x_n) \in U, v_i \in V\rangle \subseteq F(X_\infty)$ of V is the closed set of laws of $\mathfrak{U}\mathfrak{V}$.*

Proof. First, we will show that every word $w \in U(V)$ is a law in the variety $\mathfrak{U}\mathfrak{V}$. Since $U(V)$ is generated by words of the form $u(v_1, \ldots, v_n), u = u(x_1, \ldots, x_n) \in U, v_i \in V$, it suffices to show that all words of this form are laws in $\mathfrak{U}\mathfrak{V}$.

Let $C \in \mathfrak{U}\mathfrak{V}$. Then there is a group $A \in \mathfrak{U}$ such that $A \trianglelefteq C$ and $C/A \in \mathfrak{V}$. Since $C/A \in \mathfrak{V}$ and every v_i is a law in $\mathfrak{V}$, by substituting arbitrary elements from C/A into $v_i = v_i(x_{i_1}, \ldots, x_{i_{n_i}})$ we obtain

$$v_i(c_{i_1}, \ldots, c_{i_{n_i}})A = v_i(c_{i_1}A, \ldots, c_{i_{n_i}}A) = A$$

for all $c_{i_j} \in C$. Set $v_i(c^{(i)}) := v_i(c_{i_1}, \ldots, c_{i_{n_i}})$. We have that $v_i(c^{(i)}) \in A, 1 \leq i \leq n$, and hence

$$u(v_1(c^{(1)}), \ldots, v_n(c^{(n)})) = 1$$

because $A \in \mathfrak{U}$ (and u is a law in A). This shows that every group in $\mathfrak{U}\mathfrak{V}$ satisfies every law in $U(V)$.

Conversely, assume that C is a group which satisfies all laws of the form $u(v_1, \ldots, v_n)$, $u = u(x_1, \ldots, x_n) \in U, v_i \in V$. We will show that $C \in \mathfrak{U}\mathfrak{V}$.

Consider $V(C)$, the verbal subgroup of C corresponding to V. Since the verbal subgroup $V(C)$ is fully invariant in C, we have $V(C) \trianglelefteq C$. We also have that $C/V(C) \in \mathfrak{V}$ because elements in $C/V(C)$ satisfy every law $v = v(x_1, \ldots, x_k) \in V$, i.e.,

$$v(c_1V(C), \ldots, c_kV(C)) = v(c_1, \ldots, c_k)V(C) = V(C)$$

for all $c_i \in C$. It remains to show that $V(C) \in \mathfrak{U}$. For this we let $u = u(x_1, \ldots, x_n)$ be a law in $\mathfrak{U}$, and $a_1, \ldots, a_n \in V(C)$. Since V is closed, V(C) consists of all elements $v(c), v \in V, c \in C^m$ for some m (when v is a word in m variables). It follows that $a_i = v_i(c^{(i)}), c^{(i)} \in C^{m(i)}$, and

$$u(a_1, \ldots, a_n) = u(v_1(c^{(1)}), \ldots, v_n(c^{(n)})) = 1,$$

by assumption. Hence u is a law in $V(C)$, i.e., we can conclude that $V(C) \in \mathfrak{U}$.

$\square$

Note that the variety of all groups is defined by the trivial closed set $V = \{1\}$ in $F(X_\infty)$, i.e., there is no word $1 \neq v \in V$.

Remark 3.2.6. *If the closed set $V \subseteq F(X_\infty)$ is non-trivial, then V is a free group of infinite rank, or, more precisely, V is isomorphic to $F(X_\infty)$.*

Proof. We will apply the fact that every subgroup of a free group is free (cf. Theorem 5.3.2, Schreier's Theorem).

Let $F(X_\infty)$ be freely generated by $X_\infty = \{x_i \mid i \in \omega\}$, and $v = v(x_1, \ldots, x_n) \in V$ a non-trivial reduced word in n variables. Since V is a closed subset of F, we obtain that $V(F) = V$. In particular, $v(x_{i_1}, \ldots, x_{i_n}) \in V$ for any choice of pairwise distinct variables $x_{i_j} \in X_\infty$, and any variable $x_i \in X_\infty$ is present in the reduced form of some word in V. If however V were a free group of finite rank, i.e., $V = \langle w_1, \ldots, w_k \rangle$ for some $w_j \in V$, then only the finitely many variables appearing in $w_1, \ldots, w_k$ would be present in the reduced forms of words in V. This is a contradiction, hence V must be of (countably) infinite rank. □

This theorem will be crucial in Section 6.1, see [27, p. 39, Remark 21.21].

Theorem 3.2.7. *If $\mathfrak{V}$ is not the variety of all groups and $\mathfrak{U}_1\mathfrak{V} \subseteq \mathfrak{U}_2\mathfrak{V}$, then $\mathfrak{U}_1 \subseteq \mathfrak{U}_2$. In particular, $\mathfrak{U}_1\mathfrak{V} = \mathfrak{U}_2\mathfrak{V}$ implies $\mathfrak{U}_1 = \mathfrak{U}_2$.*

Proof. Since $\mathfrak{U}_1\mathfrak{V} \subseteq \mathfrak{U}_2\mathfrak{V}$, we have $U_2(V) \subseteq U_1(V)$, by Theorem 3.2.5. Then $U_2(F) \subseteq U_1(F)$, by Remark 3.2.6. Therefore $U_2 \subseteq U_1$ because U_1 and U_2 are closed. We can conclude that $\mathfrak{U}_1 \subseteq \mathfrak{U}_2$. □

3.3 Burnside varieties

The *Burnside variety* $\mathfrak{B}_n$ *of exponent* n, is the variety that is defined by the law $x^n = 1$. The free group on X in $\mathfrak{B}_n$ is denoted by $B(X, n)$. For convenience we call any group in a Burnside variety a *Burnside group* and any group $B(X, n)$ a *free Burnside group*.

In 1902 (see [3]) Burnside posed the question whether any finitely generated group in which every element is of finite order must necessarily be finite. Novikov and Adjan constructed in 1968 a counterexample in their very long and complicated papers (Infinite periodic groups I, II, III, see [28]), which established a finitely generated but infinite group for odd exponents $n \geq 4381$. Later Ol'shanskii shortened their result for odd $n > 10^{10}$ in [30].

In this section, we will study the inductive constructions by Ol'shanskii which will give us at the end an infinite group isomorphic to the free Burnside group $B(X, n)$ for a sufficiently large odd n. Moreover, by using some more effort, we obtain a particular group (infinite such that every non-trivial proper subgroup is cyclic of order p) which turns out to be a negative answer to Burnside's question as well.

We first need some auxiliary terminology for the constructions:

Recall that the free group F on X consists of reduced words constructed from letters in $X \cup X^{-1}$, where we assume that X does not contain any symbols of the form x^{-1} with $x \in X$.

Next, assume that a set of relators $\mathcal{R}$ is decomposed into subsets: Let

$$\mathcal{R} = \bigcup_{i<\omega} \mathcal{S}_i, \tag{3.1}$$

where $\mathcal{S}_i$ may be empty for some i, and if $i \neq j$, then any word in $\mathcal{S}_i \subseteq F(X)$ is neither a conjugate nor the inverse of any word in $\mathcal{S}_j$. Then the following presentation

$$G := \langle X \mid \mathcal{R} = \bigcup_{i<\omega} \mathcal{S}_i \rangle \tag{3.2}$$

together with the decomposition (3.1) is said to be a *graded presentation* and the members of $\mathcal{S}_i$ are called *relators of rank i*. We set

$$G(i) := \langle X \mid \mathcal{R}_i := \bigcup_{j=1}^{i} \mathcal{S}_j \rangle, \tag{3.3}$$

and introduce some terminology relative to $G(i)$.

Definition 3.3.1. *(i) If two given words $u, v \in F(X)$ represent the same element of $G(i)$, then we say that u and v are* equal in rank i *and write $u =^i v$.*

(ii) We say that u is a conjugate of v in rank i *if there is a word w in $G(i)$ such that $u =^i w^{-1}vw$.*

Note that if we set $\mathcal{R}_0 := \emptyset$, then $G(0) = F(X)$, and hence $u =^0 v$ means that $u = v$ in the free group $F(X)$.

There are two important properties of graded presentations involved in the constructions, asphericity and atoricity, which relate fundamental algebraic properties of the group G to geometric and topological properties of its set $\mathcal{R}$ of relators. An aspherical presentation guarantees the mutual independence of the equations $r = 1$ in the set $\{r = 1 \mid r \in \mathcal{R}\}$ of defining relations, while an atoroidal presentation implies that any two commuting elements $x, y \in G$ generate a cyclic subgroup $\langle x, y \rangle \subseteq G$.

The asphericity and atoricity of the graded presentations constructed below is established by a number of technical estimates that hold provided that n is odd and sufficiently large. In the case of the presented construction for $B(X, n)$ this will be guaranteed for all odd $n > 10^{10}$. Next, we will follow the construction given in [31, pp. 196–197].

The construction for the free Burnside group $B(X, n)$:

We start by setting $\mathcal{R}_0 = \emptyset$ and $G(0) = F(X)$ and continue constructing inductively: For $i \geq 1$, assume that the group $G(i-1)$, the set of relators $\mathcal{R}_{i-1}$ and sets of words $\mathfrak{X}_j, 0 < j < i$, have already been defined.

Definition 3.3.2. *(i) The* length *of a word w is defined by*

$$|\,w\,| := \begin{cases} 0 & \textit{if } w = 1 \\ n & \textit{if } w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}, x_i \in X, \varepsilon_i = \pm 1, \textit{as a reduced word.} \end{cases}$$

(ii) A word a is called a period in rank j *if $a \in \mathfrak{X}_j$.*

(iii) A non-trivial word w is simple in rank i *if it is not conjugate in rank i to a power of any period of rank $k \leq i$ and not conjugate in rank i to a power of any word v such that $|\,v\,| < |\,w\,|$.*

Let $\mathfrak{X}_i$ be a maximal set of reduced words of length i which are simple in rank $i-1$ such that for any distinct words u, v of $\mathfrak{X}_i$, u is not a conjugate of v or v^{-1} in rank $i-1$.

Now we enlarge the set of relators $\mathcal{R}_{i-1}$ to get $\mathcal{R}_i$ and $G(i)$ as follows:
Set $\mathcal{S}_i := \{a^n \mid a \in \mathfrak{X}_i\}$, $\mathcal{R}_i := \mathcal{R}_{i-1} \cup \mathcal{S}_i$ and define

$$G(i) := \langle X \mid \mathcal{R}_i \rangle.$$

The words in $\mathfrak{X}_i$ are the *periods in rank* i, cf. Definition 3.3.2. Finally, we put

$$G(\omega) := \langle X \mid \mathcal{R} = \bigcup_{i<\omega} \mathcal{R}_i \rangle.$$

The set of relators $\mathcal{R} = \bigcup_{i<\omega} \mathcal{R}_i$ induces the graded presentation $G(\omega) = \langle X \mid \mathcal{R} \rangle$, which is obtained as a direct limit of the sequence of canonical epimorphisms $G(i) \longrightarrow G(i+1)$. If N_i is the normal subgroup of $F(X)$

generated by $\mathcal{R}_i$, then $G(i) = F(X)/N_i$ and the groups N_i, $i < \omega$, constitute an ascending chain of normal subgroups with $N := \bigcup_{i<\omega} N_i$ and $G(\omega) = F(X)/N$.

We collect some interesting properties of the groups $G(i)$, cf. [31, pp. 198–217]. We will omit any proofs of the following results, which are based on elaborate considerations from combinatorial topology and combinatorial group theory.

Lemma 3.3.3. *The presentation $G(i) = \langle X \mid \mathcal{R}_i \rangle$ is aspherical and atoroidal.*

This implies the following

Corollary 3.3.4. *None of the relations in the set $\{r = 1 \mid r \in \mathcal{R}_i\}$ follows from the others.*

Corollary 3.3.5. *If $uv =^i vu, u, v \in F(X)$, then there is a word w such that $u =^i w^k$ and $v =^i w^l$ for some integers k and l.*

The following shows that $G(\omega)$ is a group in the Burnside variety $\mathfrak{B}_n$, and that $G(\omega)$ is a counterexample to Burnside's problem for finite $|X| > 1$.

Theorem 3.3.6. *The law $x^n = 1$ holds in the group $G(\omega)$. In addition, if $|X| > 1$, then $G(\omega)$ is an infinite group.*

We collect some significant properties of the group $G(\omega)$ in

Theorem 3.3.7. *Every abelian subgroup of $G(\omega)$ is cyclic of order diving n.*

Theorem 3.3.8. *If $|X| > 1$, then the set of relations $\{r = 1 \mid r \in \mathcal{R}\}$ of $G(\omega)$ is infinite and independent. It is not equivalent to any finite set of relations.*

Theorem 3.3.9. *Every finite subgroup of $G(\omega)$ is cyclic.*

We furthermore have a tool for representing elements of $G(\omega)$ by periods, see [31, p. 198, Lemma 18.3 and p. 214, Theorem 19.4].

Proposition 3.3.10. *For any $w \in F(X) \setminus N$, there are $k \leq i < \omega, 1 \leq m < n, g \in G(\omega) = F(X)/N, x \in \mathfrak{X}_k$ such that $w^g =^i x^m$, i.e., w is conjugate in rank i to a non-trivial power of a period $x \in \bigcup_{j=1}^{i} \mathfrak{X}_j$.*

The next theorem assures that the construction of $G(\omega)$ actually provides the free Burnside group $B(X, n)$.

Theorem 3.3.11. *$G(\omega)$ is isomorphic to the free Burnside group $B(X, n)$ in the Burnside variety $\mathfrak{B}_n$.*

Next we will adjust the way we enlarge $\mathcal{R}_{i-1}$ to $\mathcal{R}_i$ in order to construct a group with even more specific properties. We need one more definition from group theory: Let G be a group, and H and K two subgroups of G. A subset of G of the form

$$HaK := \{hak \mid h \in H, k \in K\},$$

where $a \in G$, is called a *double coset* of the pair (H, K) of subgroups of G. Double cosets define a partition of G, which is induced by the following equivalence relation:

$$a \sim b \text{ if there are } h \in H \text{ and } k \in K \text{ such that } hak = b.$$

The group constructed below will be one of our most important tools. Theorem 3.3.14 shows that this group is an element of the Burnside variety $\mathfrak{B}_n$. We denote this Burnside group by $\mathcal{B}$ and will follow the construction given in [31, pp. 270–272 and 297–298]. This construction is guaranteed for all odd $n > 10^{75}$ provided an appropriate choice of positive integers for the auxiliary parameters d, ℓ and h.

<u>The construction for the Burnside group $\mathcal{B}$</u>:

We begin by setting $X = \{x_1, x_2\}$ as the set of generators. Thus, there will be only two generators. Put $\mathcal{R}_0 = \emptyset$ and $G(0) = F(X)$. We define the graded presentation inductively: For $i \geq 1$, assume that the group $G(i-1)$, the set of relators $\mathcal{R}_{i-1}$ and sets of words $\mathfrak{X}_j, 0 < j < i$, have already been defined. As in the previous construction, let $\mathfrak{X}_i$ be a maximal set of reduced words of length i which are simple in rank $i-1$ such that for any distinct words u, v of $\mathfrak{X}_i$, u is not a conjugate of v or v^{-1} in rank $i-1$. The elements of $\mathfrak{X}_i$ will be called the periods in rank i.

Now we construct a set $\mathcal{S}_i$ of relators as follows:

- Add relators of type 1 to $\mathcal{S}_i$: If $a \in \mathfrak{X}_i$, then add a^n to $\mathcal{S}_i$. The relation

  $$a^n = 1 \tag{3.4}$$

  is called a *defining relation of type* 1 *in rank* i, while a^n is called a *relator of type* 1 *in rank* i.

- Add relators of type 2 to $\mathcal{S}_i$: For each period $a \in \mathfrak{X}_i$, we consider a maximal set $\mathfrak{Y}_a \subseteq F(X)$ such that the following holds.

  - If $t \in \mathfrak{Y}_a$, then $1 \leq |\,t\,| < d\,|\,a\,|$.
  - For each double coset of the pair $(\langle a\rangle, \langle a\rangle)$ of subgroups of G, there is at most one representative in $\mathfrak{Y}_a$, and this word is of minimal length among the words representing this double coset.

  For any period $a \in \mathfrak{X}_i$, we introduce relations of type 2 as follows:

  - If $x_1 \notin \langle a\rangle \subseteq G(i-1)$, then for each $t \in \mathfrak{Y}_a$ with $t \notin \langle a\rangle x_1 \langle a\rangle \subseteq$

$G(i-1)$ we introduce a defining relation of the form

$$x_1 a^\ell t a^{\ell+2} \dots t a^{\ell+2h-2} = 1. \tag{3.5}$$

– If $x_2 \notin \langle a \rangle \subseteq G(i-1)$ and $x_2 \notin \langle a \rangle x_1 \langle a \rangle \subseteq G(i-1)$, then for each $t \in \mathfrak{Y}_a$ with $t \notin \langle a \rangle x_2 \langle a \rangle \subseteq G(i-1)$ we introduce a defining relation of the form

$$x_2 a^{\ell+1} t a^{\ell+3} \dots t a^{\ell+2h-1} = 1. \tag{3.6}$$

The relations (3.5) and (3.6) are called *defining relations of type* 2 *in rank* i and their left hand sides are called *relators of type* 2 *in rank* i, and we add all relators of type 2 in rank i to $\mathcal{S}_i$.

Next, we set, as before, $\mathcal{R}_i := \mathcal{R}_{i-1} \cup \mathcal{S}_i$ and define

$$G(i) := \langle X \mid \mathcal{R}_i \rangle$$

and

$$G(\omega) := \langle X \mid \mathcal{R} = \bigcup_{i<\omega} \mathcal{R}_i \rangle.$$

This graded presentation exhibits similar properties as the previous construction. We mention in particular

Lemma 3.3.12. *The presentation $G(i) = \langle X \mid \mathcal{R}_i \rangle$ is aspherical and atoroidal.*

For the constructed group $\mathcal{B} = G(\omega) = \langle X \mid \mathcal{R} \rangle$, we have the following important result which identifies $\mathcal{B}$ for all primes $p := n > 10^{75}$ as an example of a so-called Tarski monster group.

Theorem 3.3.13. *Let p be a prime. Then the group $\mathcal{B}$ is an infinite group such that every non-trivial proper subgroup is cyclic of order p.*

In particular, the group $\mathcal{B}$ satisfies the law $x^p = 1$, hence it is a Burnside group in the Burnside variety $\mathfrak{B}_p$ and a counterexample to Burnside's problem. This will be applied later in Section 5.4 and 6.1.

In the more general case of odd $n > 10^{75}$, this result weakens to

Theorem 3.3.14. *The group $\mathcal{B}$ is infinite, generated by two elements and a member of the Burnside variety $\mathfrak{B}_n$. Every proper subgroup of $\mathcal{B}$ is cyclic and every maximal proper subgroup of $\mathcal{B}$ is of order n. In addition, the intersection of any two distinct subgroups of order n is trivial.*

We formulate one more useful result from [31, p. 334, Theorem 31.1], which applies to both our constructions.

Theorem 3.3.15. *Let $G = G(\omega) = \langle X \mid \mathcal{R} = \bigcup_{i<\omega} \mathcal{R}_i \rangle$ be a graded presentation such that $G(i) = \langle X \mid \mathcal{R}_i \rangle$ is an aspherical presentation for all $i < \omega$. If $G = F/N$ is the free presentation of G (with $F = F(X)$ and $N = \mathcal{R}^F$), then the following holds:*

(a) Let $w \in N$. Then the following are equivalent.

- *(i) $w \in [F, N]$.*
- *(ii) If $w = \prod_{1 \le j \le k} s_j^{-1} r_j^{\varepsilon_j} s_j$, with $r_j \in \mathcal{R}, s_j \in F$, $\varepsilon_j = \pm 1$, then $\sum_{r_j = r} \varepsilon_j = 0$ for all $r \in \mathcal{R}$.*

(b) $N/[F, N]$ is a free abelian group with basis $\{r[F, N] \mid r \in \mathcal{R}\}$.

Remark 3.3.16. *It is clear that $N/[F, N]$ is abelian, however, the asphericity of the presentations $G(i)$ implies that $\{r = 1 \mid r \in \mathcal{R}\}$ is an independent set of relations and, consequentially, that $\{r[F, N] \mid r \in \mathcal{R}\}$ is a basis of $N/[F, N]$.*

Chapter 4

Cellular covers of groups

We recall some basic facts about cellular covers of groups from [5, 6, 7, 8, 12, 13]. We will mainly follow [13].

4.1 Cellular covers of groups

We begin with definitions from [13, pp. 319–320].

Definition 4.1.1. *A (group) homomorphism $e : H \longrightarrow G$ is a* cellular cover *of G if any homomorphism $\varphi \in \mathrm{Hom}(H, G)$ factors uniquely through e, i.e., there is a unique homomorphism $\overline{\varphi} : H \longrightarrow H$ such that $\overline{\varphi}e = \varphi$.*

It is equivalent to saying that $e : H \longrightarrow G$ is a cellular cover of G if the following diagram commutes.

$$\begin{tikzcd} H \arrow[d, dotted, "\exists!\overline{\varphi}"'] \arrow[dr, "\forall\varphi"] & \\ H \arrow[r, "e"'] & G \end{tikzcd}$$

We can see that there are many non-isomorphic cellular covers of a group G. For instance, if $G = \mathbb{Q} \oplus \mathbb{Z}_2$, then both $\mathbb{Q}$ and $\mathbb{Z}_2$ define cellular covers together with the inclusion maps for these subgroups. This is because any endomorphism of G carries the divisible part $\mathbb{Q}$ of G to itself and, similarly, carries $\mathbb{Z}_2$ to $\mathbb{Z}_2$, which is the torsion part of G.

With the help of the next definitions we can assign cellular covers to a given group A.

Definition 4.1.2. *Let A, B, G, H be groups. We say that*

*(i) a homomorphism $e : H \longrightarrow G$ is A-*injective *if*

$$e_* : \mathrm{Hom}(A, H) \longrightarrow \mathrm{Hom}(A, G) \ (\overline{\varphi} \mapsto \overline{\varphi}e)$$

is injective,

*(ii) $e : H \longrightarrow G$ is A-*surjective *if e_* is surjective,*

*(iii) $e : H \longrightarrow G$ is A-*equivalent *if e_* is bijective, and*

*(iv) B is A-*cellular *if any A-equivalent homomorphism $e : H \longrightarrow G$ is also B-equivalent.*

Definition 4.1.3. *A homomorphism $e : H \longrightarrow G$ is an A-*cellular cover *of G if H is A-cellular and e is an A-equivalence.*

Every A-cellular cover is a cellular cover, but also the converse is true: Let cell G be the collection of all cellular covers of G. It was proved in [13, p. 320, Remark 2.4 and p. 325] that cell G is exactly the same as $\mathrm{cell}_A G$, the collection of A-cellular covers of G for all groups A.

Some results related to commutators, which originate with Chachólski, Damian, Farjoun, and Segev [4], are collected from [13]. We will also elaborate their proofs which the reader can find in [4], [13], and also in [17, Vol. 2, p. 746, Proposition 25.57 and p. 747, Lemma 25.58].

We need one more notation for the next step, which is called the A-*socle* of H,

$$\mathfrak{s}_A H = \langle A\sigma \mid \sigma \in \mathrm{Hom}(A, H)\rangle.$$

Proposition 4.1.4. *Let A be a group. Then*

$$e : H \longrightarrow G \text{ is } A\text{-injective } \Leftrightarrow ([\mathfrak{s}_A H, \mathrm{Ker}\, e] = 1 \text{ and } \mathrm{Hom}(A, \mathrm{Ker}\, e) = 0).$$

Remark 4.1.5. *Observe that we use the convention* $\mathrm{Hom}(A, \mathrm{Ker}\, e) = 0$ *from module theory to denote a trivial group, albeit* $\mathrm{Hom}(A, \mathrm{Ker}\, e)$ *is a non-abelian group. By* 0 *we denote the trivial zero homomorphism* $0 : A \longrightarrow 1 \subseteq \mathrm{Ker}\, e$.

Proof. Assume that $[\mathfrak{s}_A H, \mathrm{Ker}\, e] = 1$ and $\mathrm{Hom}(A, \mathrm{Ker}\, e) = 0$. To show that $e : H \longrightarrow G$ is A-injective, let $\psi_1, \psi_2 \in \mathrm{Hom}(A, H)$ with $\psi_1 e = \psi_2 e$. We will show that $\psi_1 = \psi_2$. Define

$$\psi : A \longrightarrow \mathrm{Ker}\, e \ (a \mapsto (a\psi_1)(a^{-1}\psi_2)).$$

We first need to check that $A\psi \subseteq \mathrm{Ker}\, e$ and ψ is a homomorphism:
Let a, b be arbitrary elements in A. We have that

$$((a\psi_1)(a^{-1}\psi_2))e = (a\psi_1 e)(a^{-1}\psi_2 e) = (a\psi_1 e)(a\psi_2 e)^{-1} = (a\psi_1 e)(a\psi_1 e)^{-1} = 1,$$

i.e., $A\psi \subseteq \operatorname{Ker} e$, and

$$(ab)\psi = (ab)\psi_1(ab)^{-1}\psi_2 = a\psi_1(b\psi_1 b^{-1}\psi_2)a^{-1}\psi_2 = a\psi_1 b\psi a^{-1}\psi_2.$$

Since $b\psi \in \operatorname{Ker} e$, $a^{-1}\psi_2 \in \mathfrak{s}_A H$ and by assumption $[\mathfrak{s}_A H, \operatorname{Ker} e] = 1$, we obtain that

$$(ab)\psi = a\psi_1 a^{-1}\psi_2 b\psi = a\psi b\psi.$$

Now $\psi \in \operatorname{Hom}(A, \operatorname{Ker} e)$, hence, by assumption, $\psi = 0$. Then for all $a \in A$

$$1 = a\psi = a\psi_1 a^{-1}\psi_2 = a\psi_1(a\psi_2)^{-1}.$$

This shows that $a\psi_1 = a\psi_2$ for all $a \in A$. Therefore, we can conclude that e is A-injective.

To prove the other implication, assume that $e : H \longrightarrow G$ is A-injective. We will first show that $[\mathfrak{s}_A H, \operatorname{Ker} e] = 1$. Let $d \in \operatorname{Ker} e$ and consider the inner automorphism induced by d

$$d^* : H \longrightarrow H \ (x \mapsto d^{-1}xd).$$

For all $x \in H$, using that $d \in \operatorname{Ker} e$, we have

$$xd^*e = (d^{-1}xd)e = (de)^{-1}xede = xe.$$

Hence $d^*e = e$ and $\varphi d^*e = \varphi e$ for any $\varphi \in \operatorname{Hom}(A, H)$. Thus we obtain that $\varphi d^* = \varphi$, because of the A-injectivity of e. It follows that

$$a\varphi = a\varphi d^* = d^{-1}(a\varphi)d$$

for all $a \in A$. Hence $d(a\varphi) = (a\varphi)d$ for all $a \in A$. This gives us that $[\mathfrak{s}_A H, \operatorname{Ker} e] = 1$, as desired.

Next, we will show that $\operatorname{Hom}(A, \operatorname{Ker} e) = 0$. Let $\varphi \in \operatorname{Hom}(A, \operatorname{Ker} e)$. Then $\varphi : A \longrightarrow \operatorname{Ker} e \subseteq H$, and $A\varphi e = 1$, i.e., $\varphi e = 0 = 0e$. Therefore, $\varphi = 0$ since e is A-injective. $\square$

Lemma 4.1.6. *(i) Let $e : X \longrightarrow Y$ be A-injective and $\varphi \in \mathrm{Hom}(A,Y)$ such that $\overline{\varphi}e = \varphi$ for some (uniquely) determined $\overline{\varphi} \in \mathrm{Hom}(A,X)$. Then $[\varphi^{-1}\mathfrak{z}(A\varphi), A] \subseteq \mathrm{Ker}\,\overline{\varphi}$.*

(ii) Let $e : X \longrightarrow Y$ be an A-equivalence and suppose that N is normal in A. Then $e : X \longrightarrow Y$ is an $A/[A,N]$-equivalence as well.

Proof. (i) Put $V := A\varphi$, and $U := e^{-1}(V)$, the pre-image of V under e. First we will claim that $[e^{-1}(\mathfrak{z}V), \mathfrak{s}_A U] = 1$.

Let $z \in e^{-1}(\mathfrak{z}V)$ and $\psi \in \mathrm{Hom}(A,U)$. Using the inner automorphism z^* induced by z on X, we obtain for all $a \in A$ that

$$a(\psi z^* e) = (a\psi)z^* e = (z^{-1}(a\psi)z)e = (ze)^{-1}(a\psi e)(ze).$$

Since $ze \in \mathfrak{z}V$ and $a\psi e \in V$, we see that $\psi z^* e = \psi e$. By A-injectivity of e, we can further conclude that $\psi z^* = \psi$. It follows $a\psi = a\psi z^* = z^{-1}(a\psi)z$, i.e., $z(a\psi) = (a\psi)z$. This completes the proof of the claim.

Since $\overline{\varphi}e = \varphi$, we have $A\overline{\varphi} \subseteq U$, hence, by definition of A-socle, $A\overline{\varphi} \subseteq \mathfrak{s}_A U$. Moreover, $(\varphi^{-1}\mathfrak{z}V)\varphi = \mathfrak{z}V$ implies $(\varphi^{-1}\mathfrak{z}V)\overline{\varphi}e = \mathfrak{z}V$. It follows that $(\varphi^{-1}\mathfrak{z}V)\overline{\varphi} \subseteq e^{-1}(\mathfrak{z}V)$. Applying the last two inclusions and the claim, we then can conclude that

$$[\varphi^{-1}\mathfrak{z}(A\varphi), A]\overline{\varphi} = [(\varphi^{-1}\mathfrak{z}(A\varphi))\overline{\varphi}, A\overline{\varphi}] \subseteq [e^{-1}(\mathfrak{z}V), \mathfrak{s}_A U] = 1,$$

which proves (i).

(ii) First, we will show that e is $A/[A,N]$-surjective.

Let $\widehat{\varphi} \in \mathrm{Hom}(A/[A,N], Y)$. We must find $\varphi' \in \mathrm{Hom}(A/[A,N], X)$ such that $\varphi' e = \widehat{\varphi}$.

Let $\pi : A \longrightarrow A/[A,N]$ be the canonical projection. Then $\varphi := \pi\widehat{\varphi} \in \mathrm{Hom}(A,Y)$. By A-surjectivity of e, there is $\overline{\varphi} \in \mathrm{Hom}(A,X)$ with $\overline{\varphi}e = \varphi$.

Note that $N\varphi \subseteq A\varphi$ and $[A\varphi, N\varphi] = [A, N]\varphi = [A, N]\pi\widehat{\varphi} = 1$. Thus we obtain $N\varphi \subseteq \mathfrak{z}(A\varphi)$. That is $N \subseteq \varphi^{-1}\mathfrak{z}(A\varphi)$. Applying this inclusion and (i), we see that

$$[A, N] \subseteq [A, \varphi^{-1}\mathfrak{z}(A\varphi)] \subseteq \operatorname{Ker}\overline{\varphi}.$$

So, $\overline{\varphi}$ induces the homomorphism

$$\varphi' : A/[A, N] \longrightarrow X \ (a[A, N] \mapsto a\overline{\varphi})$$

with

$$(a[A, N])\varphi' e = a\overline{\varphi}e = a\varphi = a\pi\widehat{\varphi} = (a[A, N])\widehat{\varphi}.$$

Hence, e is $A/[A, N]$-surjective.

It remains to show that e is $A/[A, N]$-injective.

Let $\varphi_1, \varphi_2 \in \operatorname{Hom}(A/[A, N], X)$ such that $\varphi_1 e = \varphi_2 e$. We will show that $\varphi_1 = \varphi_2$. We again use the projection π, and obtain that $\pi\varphi_1 e = \pi\varphi_2 e$.

$$\begin{tikzcd}
 & A \arrow[d, "\pi"] & \\
 & A/[A, N] \arrow[dl, "\varphi_1, \varphi_2"'] \arrow[dr, "\varphi_1 e = \varphi_2 e"] & \\
X \arrow[rr, "e"'] & & Y
\end{tikzcd}$$

Since e is A-injective, it follows that $\pi\varphi_1 = \pi\varphi_2$. Thus

$$(a[A, N])\varphi_1 = a\pi\varphi_1 = a\pi\varphi_2 = (a[A, N])\varphi_2.$$

Hence, e is also $A/[A, N]$-injective. Therefore, e is $A/[A, N]$-equivalent. $\square$

We obtain the following consequence, which is also given in [17, Vol. 2, p. 748, Corollary 25.59].

Corollary 4.1.7. *Let N be a normal subgroup of G and G an A-cellular group. Then $G/[G, N]$ is A-cellular as well.*

Proof. Let $e : X \longrightarrow Y$ be A-equivalent. We will show that e is also $G/[G, N]$-equivalent. Since G is A-cellular and e is A-equivalent, we obtain that e is G-equivalent as well. Applying Lemma 4.1.6 (ii), we can conclude that e is $G/[G, N]$-equivalent. $\square$

We call $\{G_\alpha \subseteq G \mid \alpha \leq \lambda\}$, λ an ordinal, an *A-chain* of G, if the following holds:

(i) $G_0 = 1$.

(ii) If $\alpha \leq \lambda$, then G_α is normal in G.

(iii) If $\alpha \leq \lambda$ is a limit ordinal, then $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$.

(iv) If $\alpha < \lambda$, then $G_\alpha \subsetneq G_{\alpha+1}$ and $G_{\alpha+1}/G_\alpha$ is an epimorphic image of A.

If the chain cannot be extended any further, or equivalently, $\mathrm{Hom}(A, G/G_\lambda) = 0$, then we call G_λ the *hyper-A subgroup* of G. This group is independent of the choice of the A-chain and is denoted by $\mathfrak{h}_A G := G_\lambda$. We say that a group G is *hyper-A* if $\mathfrak{h}_A G = G$, i.e., if every non-trivial epimorphic image of G contains a non-trivial epimorphic image of A.

The next proposition is the analogue of Corollary 4.1.7 for hyper-A groups. Its proof extends ideas in [13, p. 323, Proposition 3.7].

Proposition 4.1.8. *If G is A-cellular and N is hyper-A and a normal subgroup of G, then G/N is A-cellular as well.*

Proof. Let $e : X \longrightarrow Y$ be A-equivalent. We will show that $e : X \longrightarrow Y$ is G/N-equivalent, i.e., G/N-surjective and G/N-injective.

Put $K = \mathrm{Ker}\, e$. Since $e : X \longrightarrow Y$ is A-injective, by Proposition 4.1.4, we obtain that $\mathrm{Hom}(A, K) = 0$.

First, we claim that $\operatorname{Hom}(N, K) = 0$. Suppose that $0 \neq \alpha \in \operatorname{Hom}(N, K)$. Then $N\alpha \neq 1$, and $N\alpha \cong N/\operatorname{Ker}\alpha$, by the first homomorphism theorem. Applying that N is hyper-A, $N\alpha$ contains a non-trivial epimorphic image of A, say A/M. Thus $A/M \subseteq N\alpha \subseteq K$. We obtain a homomorphism

$$0 \neq \beta : A \longrightarrow A/M \subseteq K.$$

It follows that $0 \neq \beta \in \operatorname{Hom}(A, K) = 0$, a contradiction. This proves the claim.

Next, we will show that $e : X \longrightarrow Y$ is G/N-surjective. Let $\psi \in \operatorname{Hom}(G/N, Y)$, and consider the canonical projection $\pi : G \longrightarrow G/N$. We have $\varphi := \pi\psi : G \longrightarrow Y$. Since G is A-cellular, $e : X \longrightarrow Y$ is G-surjective. Thus there is $\overline{\varphi} \in \operatorname{Hom}(G, X)$ such that $\overline{\varphi}e = \varphi$. We then have

$$N\overline{\varphi}e = N\varphi = N\pi\psi = 1\psi = 1.$$

Hence $N\overline{\varphi} \subseteq \operatorname{Ker} e = K$. By the claim, $\overline{\varphi}$ is the zero map from N to K, hence $N\overline{\varphi} = 1$. Therefore, the homomorphism $\overline{\varphi} : G \longrightarrow X$ induces the homomorphism

$$\overline{\psi} : G/N \longrightarrow X \ (gN \mapsto g\overline{\varphi})$$

such that

$$(gN)\overline{\psi}e = g\overline{\varphi}e = g\varphi = g\pi\psi = (gN)\psi$$

for all $g \in G$. Hence $e : X \longrightarrow Y$ is G/N-surjective, as required.

To show that $e : X \longrightarrow Y$ is G/N-injective, replace A by G and $A/[A, N]$ by G/N in the proof of Lemma 4.1.6 (ii). $\square$

The next theorem guarantees the uniqueness of an A-cellular cover of a group G for any pair of groups A and G, and it also sheds some light on the construction and group structure of this A-cellular cover. We elaborate the proof given in [13, p. 331, Theorem 6.3] in the following

Theorem 4.1.9. *Let $\mathfrak{V}$ be a variety, $G \in \mathfrak{V}$ and A an arbitrary group. Then the following holds:*

(i) There is a unique A-cellular cover $e : H \longrightarrow G$.

(ii) The group H in (i) is a central extension by a subgroup of G. Thus

$$1 \longrightarrow \operatorname{Ker} e \longrightarrow H \longrightarrow \operatorname{Im} e \longrightarrow 1$$

is a short exact sequence with $\operatorname{Ker} e \subseteq \mathfrak{z}H$ and $\operatorname{Im} e \subseteq G$.

Proof. (i) Existence part: Let $I := \operatorname{Hom}(A, G)$ be an indexing set and consider each $\varphi \in I$ as a homomorphism $\varphi : A_\varphi \longrightarrow G$ from a distinct copy A_φ of A to G. Next, put $\overline{H} := *_{\varphi \in I} A_\varphi$, the free product of the groups A_φ. By definition of free product, we obtain a homomorphism $\overline{e} : \overline{H} \longrightarrow G$ such that $\overline{e} \restriction A_\varphi = \varphi$ for all $\varphi \in I$ (see Definition 2.1.10). Since the restriction of $\overline{e}$ to A_φ coincides with φ for all $\varphi \in I$, we see that

$$\overline{e} : \overline{H} \longrightarrow G \text{ is } A\text{-surjective.}$$

Set $\overline{K} := \operatorname{Ker} \overline{e}$. Then $[\overline{H}, \overline{K}] \trianglelefteq \overline{H}$ and $[\overline{H}, \overline{K}] \subseteq \overline{K}$, and $\overline{e}$ induces $e' : H' \longrightarrow G$ where $H' := \overline{H}/[\overline{H}, \overline{K}]$. We see that

$$e' : H' \longrightarrow G \text{ is } A\text{-surjective}$$

from the following diagram:

$$\begin{tikzcd}
& A \arrow[dr, "\varphi"] \arrow[ddl, dotted, "\overline{\varphi}"'] & \\
& H' \arrow[r, "e'"] & G \\
\overline{H} \arrow[ur, "\pi"] \arrow[urr, "\overline{e}"'] & &
\end{tikzcd}$$

For any $\varphi \in \mathrm{Hom}(A, G)$, there is $\overline{\varphi} \in \mathrm{Hom}(A, \overline{H})$ such that $\overline{\varphi}\,\overline{e} = \varphi$, since $\overline{e}$ is A-surjective. Then $\overline{\varphi}\pi \in \mathrm{Hom}(A, H')$ with $(\overline{\varphi}\pi)e' = \overline{\varphi}(\pi e') = \overline{\varphi}\,\overline{e} = \varphi$, where π is the canonical projection from $\overline{H}$ onto $H' = \overline{H}/[\overline{H}, \overline{K}]$.

Note that $N := \mathrm{Ker}\, e' = \overline{K}/[\overline{H}, \overline{K}] \subseteq \mathfrak{z}(\overline{H}/[\overline{H}, \overline{K}]) = \mathfrak{z}H'$. Let $M = \mathfrak{h}_A N$. By definition, we then obtain that $\mathrm{Hom}(A, N/M) = 0$.

Next we can consider $H := H'/M$ as $M \subseteq N \subseteq \mathfrak{z}H'$. Since $M \subseteq N$, $e' : H' \longrightarrow G$ induces a homomorphism

$$e : H \longrightarrow G, \text{ which is } A\text{-surjective as well,}$$

with $\mathrm{Ker}\, e = N/M$. We will show that

(a) H is A-cellular, and

(b) $e : H \longrightarrow G$ is A-injective.

(a): Assume that $f : X \longrightarrow Y$ is an arbitrary A-equivalence. Then, we know that

$$f_* : \mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(A, Y)\ (\overline{\varrho} \mapsto \overline{\varrho} f)$$

is bijective. It follows a bijection

$$\prod_{\varphi\in I}\operatorname{Hom}(A_\varphi, X) \longrightarrow \prod_{\varphi\in I}\operatorname{Hom}(A_\varphi, Y)\ ((\overline{\varrho_\varphi})_{\varphi\in I} \mapsto (\overline{\varrho_\varphi} f)_{\varphi\in I}).$$

But $\operatorname{Hom}(*_{\varphi\in I}A_\varphi, X) \cong \prod_{\varphi\in I}\operatorname{Hom}(A_\varphi, X)$ holds canonically (i.e., $\sigma \mapsto (\sigma_\varphi)_{\varphi\in I}$ where $\sigma_\varphi = \sigma \restriction A_\varphi$). Thus the bijection

$$\operatorname{Hom}(*_{\varphi\in I}A_\varphi, X) \longrightarrow \operatorname{Hom}(*_{\varphi\in I}A_\varphi, Y)\ (\sigma \mapsto \sigma f)$$

implies that $f : X \longrightarrow Y$ is $*_{\varphi\in I}A_\varphi = \overline{H}$-equivalent; in other words, $\overline{H}$ is A-cellular. By Corollary 4.1.7, we have that $\overline{H}/[\overline{H}, \overline{K}] = H'$ is A-cellular. To conclude that $H(= H'/M)$ is A-cellular, we apply Proposition 4.1.8.

(b): From $N \subseteq \mathfrak{z}H'$, we obtain that $\operatorname{Ker} e\ (= N/M) \subseteq \mathfrak{z}(H'/M) = \mathfrak{z}H$. Thus $[H, \operatorname{Ker} e] = 1$ and it follows that $[\mathfrak{s}_A H, \operatorname{Ker} e] = 1$, since $\mathfrak{s}_A H \subseteq H$. Moreover, we have that $\operatorname{Hom}(A, \operatorname{Ker} e) = \operatorname{Hom}(A, N/M) = 0$. Hence, by Proposition 4.1.4, we can conclude that $e : H \longrightarrow G$ is A-injective.

Uniqueness part: Let $e_i : H_i \longrightarrow G$ $(i = 1, 2)$ be two A-cellular covers of G. Applying that $e_2 : H_2 \longrightarrow G$ is A-equivalent and H_1 is A-cellular, we obtain a unique homomorphism $\eta_1 : H_1 \longrightarrow H_2$ such that $\eta_1 e_2 = e_1$.

$$\begin{tikzcd}
H_1 \arrow[d, dashed, "\eta_1"'] \arrow[dr, "e_1"] & \\
H_2 \arrow[r, "e_2"'] & G
\end{tikzcd}$$

Using the same argument for the A-equivalence $e_1 : H_1 \longrightarrow G$ and the A-cellular group H_2, we obtain a homomorphism $\eta_2 : H_2 \longrightarrow H_1$ such that $\eta_2 e_1 = e_2$. Then

$$(\eta_1\eta_2)e_1 = \eta_1(\eta_2 e_1) = \eta_1 e_2 = e_1.$$

Since $e_1 : H_1 \longrightarrow G$ is also H_1-equivalent, we can conclude further that $\eta_1\eta_2 = \mathrm{id}_{H_1}$.

$$\begin{tikzcd} H_1 \arrow[d, "\eta_1\eta_2, \mathrm{id}_{H_1}"'] \arrow[dr, "e_1"] & \\ H_1 \arrow[r, "e_1"'] & G \end{tikzcd}$$

Similarly, we get $\eta_2\eta_1 = \mathrm{id}_{H_2}$. Hence, $\eta := \eta_1$ is an isomorphism such that $\eta e_2 = e_1$ and $H_1 \cong H_2$.

(ii) follows at once from (i). □

Let $\mathfrak{A}$ be the variety of all groups. If $\mathfrak{V}$ is a variety of groups, then let

$$\text{cell } \mathfrak{V} = \langle C \mid C \text{ is an } A\text{-cellular cover of some } G \in \mathfrak{V} \text{ for some } A \in \mathfrak{A} \rangle$$

be the variety generated by all groups C satisfying the above condition. For any group $G \in \mathfrak{V}$, it is easy to see that the map $\mathrm{id} : G \longrightarrow G$ is a G-cellular cover of G. This shows that $\mathfrak{V} \subseteq \text{cell } \mathfrak{V}$ for any variety $\mathfrak{V}$. It was shown in [7, p. 62, Theorem 1.4] that cellular covers of nilpotent groups of class n are again nilpotent of class n, i.e., cell $\mathcal{N}_n \subseteq \mathcal{N}_n$, where $\mathcal{N}_n$ is the variety of nilpotent groups of class n. Hence, cell $\mathcal{N}_n = \mathcal{N}_n$, strictly speaking, the variety $\mathcal{N}_n$ is closed under taking cellular covers. However, the Burnside variety $\mathfrak{B}_p$ defined by the law $x^p = 1$ is not closed under taking cellular covers for $p > 10^{75}$ because cell $\mathfrak{B}_p$ contains all abelian groups (see [13, pp. 328–330]). Thus, it is natural to determine which varieties are closed or not closed under taking cellular covers. Some results on the latter case will be shown in Chapters 5 and 6. In order to study the first case, we need an appropriate closure of $\mathfrak{V}$ that will cover all cellular covers of all its members. The central-closure operator suggested in [13, p. 330] is an option for further developments.

Chapter 5

Burnside varieties neither cellular closed nor finitely based

In this chapter, we will construct countably many varieties which are neither cellular closed nor finitely based.

5.1 An important Example

For an arbitrary field F consider the set G_n of all matrices A of the form

$$A = \begin{pmatrix} 1 & \phi(x) & f(x,y) \\ 0 & 1 & \psi(y) \\ 0 & 0 & 1 \end{pmatrix},$$

where $\phi(x) = \sum_{i=1}^{n} a_i x_i$ and $\psi(y) = \sum_{i=1}^{n} b_i y_i$ are arbitrary linear forms and $f(x,y) = \sum_{i,j=1}^{n} c_{ij} x_i y_j$ is a bilinear form over the n-dimensional vector space $V = F^n$. We shortly write $A = \| \phi, \psi, f \|$.

Note that A is an upper triangular matrix, that the product of two upper triangular matrices is again an upper triangular matrix and that G_n is closed under matrix multiplication. Moreover, we see that A can be written as the sum of the *identity matrix* E and a nilpotent matrix, say $A = E + D$, where $D^3 = 0$, thus we can find the inverse of A from the formula $A^{-1} = E - D + D^2$. Therefore, G_n is a multiplicative group.

Define $\sigma : G_n \longrightarrow V^* \times V^*$, where $V^* = \mathrm{Hom}_F(V, F)$ is the *dual space* of V, by mapping

$$\| \phi, \psi, f \| \mapsto (\phi, \psi).$$

Let $A = \| \phi_1, \psi_1, f_1 \|$ and $B = \| \phi_2, \psi_2, f_2 \|$. Since

$$A \cdot B = \| \phi_1 + \phi_2, \psi_1 + \psi_2, f_1 + \phi_1\psi_2 + f_2 \|,$$

we have that

$$\sigma(A \cdot B) = (\phi_1 + \phi_2, \psi_1 + \psi_2) = (\phi_1, \psi_1) + (\phi_2, \psi_2) = \sigma(A) + \sigma(B),$$

and σ is a homomorphism.

Put $\mathrm{Ker}\,\sigma := H_n$. By the isomorphism theorem, we obtain that G_n/H_n is abelian. Then, by Theorem 2.1.1 follows that G_n' is contained in the kernel H_n. It is easy to see that H_n consists of matrices of the form $\| 0, 0, f \|$. We can also verify that

- $\| 0, 0, f \| \cdot \| 0, 0, f' \| = \| 0, 0, f + f' \|$,
- $\| \phi, \psi, f \|^{-1} = \| -\phi, -\psi, \phi\psi - f \|$,
- $[\| \phi, 0, 0 \|, \| 0, \psi, 0 \|] = \| -\phi, 0, 0 \| \cdot \| 0, -\psi, 0 \| \cdot \| \phi, 0, 0 \| \cdot \| 0, \psi, 0 \| = \| 0, 0, \phi\psi \|$.

Next, we will show that $\phi\psi$ is a bilinear form of rank ≤ 1:

Let $\phi(x) = \sum_{i=1}^{n} a_i x_i$ and $\psi(y) = \sum_{i=1}^{n} b_i y_i$. Then

$$\begin{aligned}
\phi(x)\psi(y) &= \sum_{i=1}^{n} a_i x_i \cdot \sum_{i=1}^{n} b_i y_i \\
&= \sum_{i,j=1}^{n} a_i b_j x_i y_j \\
&= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.
\end{aligned}$$

We can see immediately that the coefficient matrix has rank ≤ 1, i.e., $\phi\psi$ is a *bilinear form of rank* ≤ 1. Furthermore, every bilinear form f of rank ≤ 1 can evidently be written as such a product $\phi\psi$ of linear forms.

Any matrix A in H_n is of the form $\| 0, 0, f \|$, where f is bilinear of rank $\leq n$. Since every bilinear form of rank $\leq n$ can be written as a sum of at most n bilinear forms of rank 1, we obtain that

$$f = f_1 + f_2 + \cdots + f_m = \phi_1\psi_1 + \phi_2\psi_2 + \cdots + \phi_m\psi_m$$

where $m \leq n$, f_i is a bilinear form of rank 1, and ϕ_i and ψ_i are linear forms, for all $i \in \{1, \ldots, m\}$. Hence,

$$\begin{aligned}
A &= \| 0, 0, f \| \\
&= \| 0, 0, \phi_1\psi_1 + \phi_2\psi_2 + \cdots + \phi_m\psi_m \| \\
&= \| 0, 0, \phi_1\psi_1 \| \| 0, 0, \phi_2\psi_2 \| \cdots \| 0, 0, \phi_m\psi_m \| \\
&= \prod_{i=1}^{m} [\|\phi_i, 0, 0\|, \|0, \psi_i, 0\|] \in G'_n.
\end{aligned}$$

This shows that H_n is contained in G'_n. Therefore, we can conclude $H_n = G'_n$.

Next, consider the commutator of two arbitrary matrices $\|\phi_1, \psi_1, f_1\|$ and $\|\phi_2, \psi_2, f_2\|$ in G_n. We have

$$\begin{aligned}
&[\|\phi_1, \psi_1, f_1\|, \|\phi_2, \psi_2, f_2\|] \\
&= \|-\phi_1, -\psi_1, \phi_1\psi_1 - f_1\|\|-\phi_2, -\psi_2, \phi_2\psi_2 - f_2\|\|\phi_1, \psi_1, f_1\|\|\phi_2, \psi_2, f_2\| \\
&= \|0, 0, \phi_1\psi_2 - \phi_2\psi_1\|. \qquad (5.1)
\end{aligned}$$

The rank of $\phi_1\psi_2 - \phi_2\psi_1$ is at most 2, since we know that the rank of the sum of two matrices does not exceed the sum of their ranks. Then, when we consider a product of fewer than $\frac{n}{2}$ commutators in G_n, the result never reaches rank n. Therefore, the element $\|0, 0, h\| \in G'_n$, where h is a bilinear form of rank n, cannot be written as any product of less than $\frac{n}{2}$ commutators in G_n. Moreover, any power of a product of fewer than $\frac{n}{2}$ commutators is once again a product of fewer than $\frac{n}{2}$ commutators:

$$\begin{aligned}
\left(\prod_{i=1}^{m}[\|\phi_{1i}, \psi_{1i}, f_{1i}\|, \|\phi_{2i}, \psi_{2i}, f_{2i}\|]\right)^k &= \left(\prod_{i=1}^{m}\|0, 0, \phi_{1i}\psi_{2i} - \phi_{2i}\psi_{1i}\|\right)^k \\
&= \|0, 0, \sum_{i=1}^{m}(\phi_{1i}\psi_{2i} - \phi_{2i}\psi_{1i})\|^k \\
&= \|0, 0, \sum_{i=1}^{m}k(\phi_{1i}\psi_{2i} - \phi_{2i}\psi_{1i})\| \\
&= \prod_{i=1}^{m}[\|k\phi_{1i}, \psi_{1i}, f_{1i}\|, \|k\phi_{2i}, \psi_{2i}, f_{2i}\|].
\end{aligned}$$

From equation (5.1), we have $[g', g] = \|0, 0, 0\| = E$ for all $g' \in G'_n$ and $g \in G_n$. This gives us that G_n has a lower central series which stabilizes at the trivial subgroup after 2 steps.

This group has some more interesting properties; over the field $F = \mathbb{Z}_p$, where $p \geq 3$ is a prime, we can show that it is an example of an element in the

Burnside variety $\mathfrak{B}_p$ defined by the law $x^p = 1$. By the binomial coefficient theorem we can express A^p for any matrix $A = E + D$ in G_n by

$$A^p = (E+D)^p = E + \binom{p}{1} D + \binom{p}{2} D^2 + \cdots + \binom{p}{p-1} D^{p-1} + D^p.$$

But D is a nilpotent matrix of class 3 and $\binom{p}{1} \equiv \binom{p}{2} \equiv 0 \pmod{p}$. Hence A^p becomes the identity matrix. It follows that G_n satisfies the law defining the above variety, i.e., G_n belongs to $\mathfrak{B}_p$.

We summarize the properties of G_n as

Proposition 5.1.1. *Let F be a field and G_n the group of matrices A as defined above. Then the following holds:*

(i) *G_n is a nilpotent group of class 2.*

(ii) *If F is the field $\mathbb{Z}_p$ with p elements, $p \geq 3$, then G_n is a finite Burnside group of exponent p.*

(iii) *G'_n has elements which are not the products of $< \frac{n}{2}$ commutators.*

(iv) *Any power of a product of $< \frac{n}{2}$ commutators is once again a product of $< \frac{n}{2}$ commutators.*

5.2 Ol'shanskii's test groups for finding varieties which are not finitely based

In this section we will consider a group K of exponent p^2 for any prime $p > 10^{10}$ to obtain a variety as in the title.

We first consider the free Burnside group $B(X_\infty, p)$, where $X_\infty = \{x_i \mid i \in \omega\}$ is a free generating set. The following results elaborate the proofs in [31, pp. 341–343, Lemma 31.3, Lemma 31.4 and Theorem 31.6] and deal with the infinite system of laws

$$[x_1, x_2]^p = 1, ([x_1, x_2][x_3, x_4])^p = 1, \dots, ([x_1, x_2] \dots [x_{2k-1}, x_{2k}])^p = 1, \dots. \tag{5.2}$$

Lemma 5.2.1. *Let* $w_k := [x_1, x_2] \dots [x_{2k-1}, x_{2k}] \in F = F(X_\infty)$*. Then* $w_k \notin w_{k-1}^m B(X_\infty, p)$*, the set of values of* w_{k-1}^m *in* $B(X_\infty, p)$*, for all* $m \in \mathbb{Z}, k \geq 2$*.*

Proof. We consider the commutator subgroup G'_{2k} of the group G_{2k} over $\mathbb{Z}_p$. By Proposition 5.1.1 (iii) and (iv), there is an element

$$g = [g_1, g_2] \dots [g_{2k-1}, g_{2k}] \in G'_{2k}$$

such that

$$g \in w_k G_{2k} \setminus w_{k-1}^m G_{2k} \tag{5.3}$$

for all $m \in \mathbb{Z}$. Since G_{2k} is a Burnside group of exponent p and $B(X_\infty, p)$ is a free Burnside group, there exists some homomorphism $\alpha : B(X_\infty, p) \to G_{2k}$ such that $x_i\alpha = g_i, 1 \leq i \leq 2k$.

Suppose that $w_k \in w_{k-1}^m B(X_\infty, p)$. Then $w_k\alpha \in w_{k-1}^m G_{2k}$ and

$$w_k\alpha = [g_1, g_2] \dots [g_{2k-1}, g_{2k}] = g.$$

Thus, $g \in w_{k-1}^m G_{2k}$, which contradicts (5.3). This completes the proof. □

Let $G = B(X_\infty, p)$ be the free Burnside group of exponent p. In what follows, we will also make use of the results from the construction of the free Burnside group in Section 3.3. Thus, we let further $G = F/N$ be a free presentation of G such that $F = F(X_\infty)$ is freely generated by $X_\infty = \{x_i \mid i \in \omega\}$ and $N = \mathcal{R}^F$ is the normal closure of the set of relators $\mathcal{R}$ (see the construction).

Next, we consider the word $w_k = [x_1, x_2] \dots [x_{2k-1}, x_{2k}] \in F(X_\infty)$. By Lemma 5.2.1, we have $w_k \in F(X_\infty) \setminus N$, and, by Proposition 3.3.10, there are $j \leq i < \omega, 1 \leq m < p, g \in G, c \in \mathfrak{X}_j$ such that $w_k^g =^i c^m$. Note that the equality of w_k^g and c^m in $G(i)$ implies equality in G. We can see that $c^m = w_k^g$ also belongs to $w_k G$, the set of values of $[x_1, x_2] \dots [x_{2k-1}, x_{2k}]$ in G, because of the property $w_k^g = [x_1^g, x_2^g] \dots [x_{2k-1}^g, x_{2k}^g]$. Quite similarly, by Lemma 5.2.1, c is not a value in G of any word of the form $([x_1, x_2] \dots [x_{2k-3}, x_{2k-2}])^l$ for all $l \in \mathbb{Z}$.

Using temporarily additive notation for abelian groups, by Theorem 3.3.15, we obtain that $N/[F, N] = \bigoplus_{r \in \mathcal{R}} \langle r[F, N] \rangle$ is a free abelian group. With $\mathfrak{X} := \bigcup_{i<\omega} \mathfrak{X}_i$ we have $\mathcal{R} = \{x^p \mid x \in \mathfrak{X}\}$, hence $N/[F, N] = \bigoplus_{x \in \mathfrak{X}} \langle x^p[F, N] \rangle$. We first write the free abelian group $N/[F, N]$ as a direct sum

$$N/[F, N] = \langle c^p[F, N] \rangle \oplus \bigoplus_{x \in \mathfrak{X} \setminus \{c\}} \langle x^p[F, N] \rangle$$

and then consider the subgroup $L/[F, N]$ of $N/[F, N]$ defined as

$$L/[F, N] := \langle c^{p^2}[F, N] \rangle \oplus \bigoplus_{x \in \mathfrak{X} \setminus \{c\}} \langle x^p[F, N] \rangle.$$

Observe that $N/[F, N] \subseteq \mathfrak{z}(F/[F, N])$ and thus $L \trianglelefteq F$.

Setting $D := N/L$, we then have that D is a cyclic group of order p. Next, put $K := F/L$. Then $K/D \cong G$, $D = N/L \subseteq \mathfrak{z}(F/L) = \mathfrak{z}K$ as $[F, N] \subseteq L$, and K is a central extension of D by G. Moreover, observe that $x^p \in L$ for all $c \neq x \in \mathfrak{X}$, and $c^{p^2} \in L$, while $c^p \notin L$, i.e.,

$$x^p = 1_K, x \neq c, \quad c^{p^2} = 1_K \quad \text{and} \quad c^p \neq 1_K$$

hold in K.

Lemma 5.2.2. *The group K satisfies the following conditions:*

(i) $K \in \mathfrak{B}_{p^2}$.

(ii) The law $([x_1, x_2] \dots [x_{2k-3}, x_{2k-2}])^p = 1$ *holds in* K.

(iii) The law $([x_1, x_2] \dots [x_{2k-1}, x_{2k}])^p = 1$ *does not hold in* K.

Proof. (i) It is easy to see that the law $x^{p^2} = 1$ holds in K by using that $K/D \cong G$, where both G and D are of exponent p.

(ii) Let $y_1, \dots, y_{2k-2} \in F$ be representatives for arbitrary elements from K, and put

$$w := [y_1, y_2] \dots [y_{2k-3}, y_{2k-2}].$$

Note that $D = N/L \subseteq F/L = K$. We thus will distinguish between two cases:

If $w \in N$, then $w \in D$ and $w^p = 1_K$ in K as D is of order p. Thus (ii) follows.

If $w \in F \setminus N$, then w represents a non-trivial element in $G = F/N$. Thus we may write $w^g = a^s$ in G for some period $a \in \mathfrak{X}$, $g \in G$ and $1 \leq s < p$. Since $(s, p) = 1$, we obtain $a = (w^g)^t$ in G for some integer t. Because of the form of w and the choice of c, we know that $a \neq c$, which implies $a^p = 1_K$ in K. Using that $w^g = a^s$ in $G = F/N$, we have $w^g = a^s d$ in F for some $d \in N$. Applying that d represents an element of $D \subseteq \mathfrak{z}K$ in K, we can conclude that

$$(w^p)^g = (w^g)^p = (a^s d)^p = a^{sp} d^p = (a^p)^s d^p = 1_K 1_K = 1_K$$

and

$$w^p = (1_K)^{g^{-1}} = 1_K$$

in K. Hence (ii) also holds in this case, and (ii) is shown.

(iii) As mentioned before, c^m with $c \in \mathfrak{X}$, $1 \leq m < p$ is a value of $[x_1, x_2] \dots [x_{2k-1}, x_{2k}]$ in $G = F/N$. Thus we may write $w' = c^m d'$ in F, for some $d' \in N$, where $w' = [z_1, z_2] \dots [z_{2k-1}, z_{2k}], z_i \in F$. Hence $(w')^p = (c^m d')^p = c^{mp}(d')^p = c^{mp} \neq 1_K$ in K because $p \leq mp < p^2$. Hence, the law $([x_1, x_2] \dots [x_{2k-1}, x_{2k}])^p = 1$ does not hold in K. □

This result implies the following

Theorem 5.2.3. *The system of group laws* (5.2) *is not equivalent to a finite system of group laws.*

Proof. By Lemma 5.2.2 we know that the law $([x_1, x_2] \dots [x_{2k-1}, x_{2k}])^p = 1$ is not a consequence of its predecessors for all $k \geq 2$. Then the theorem follows by applying Corollary 3.1.6. □

We obtain varieties which are not finitely based by considering the variety generated by the system of group laws (5.2) for any prime $p > 10^{10}$. Therefore, we can conclude

Theorem 5.2.4. *There exist countably many pairwise distinct varieties which are not finitely based.*

5.3 A survey on cohomology groups and the Schur multiplier

Cohomology groups, homology groups and the Schur multiplier are central tools of this work. Hence we will first discuss their important basic properties and connections. Our main references are [10, 18, 22, 31] and the classical main result is Theorem 5.3.4. We will write homomorphisms on the left hand side in this section.

Let G and A be multiplicatively written groups where A is abelian. We say that G *acts on* A if for each $g \in G$ and $a \in A$ there is a unique element of A denoted by $a \cdot g$ such that

$$\begin{aligned}(ab) \cdot g &= (a \cdot g)(b \cdot g)\\ (a \cdot g) \cdot h &= a \cdot (gh)\\ a \cdot 1 &= a\end{aligned}$$

for all $a, b \in A$, $g, h \in G$ and $1 = 1_G \in G$.

We consider a function f from G^n to A. Following Eilenberg and MacLane [10] this function is called an *n-cochain* of G in A. The set of all n-cochains, denoted by $C^n(G, A)$, is an abelian group under the multiplication of values. We set $C^0(G, A) = A$, and assume that a fixed action of G on A is given. Then for $f \in C^n(G, A)$ we define a homomorphism

$$d_{n+1} \ : \ C^n(G, A) \longrightarrow C^{n+1}(G, A)$$

by the formula

$$\begin{aligned}(d_{n+1}f)(g_1, \ldots, g_{n+1}) &:= f(g_2, \ldots, g_{n+1})\\ &\times \prod_{i=1}^{n} f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1})^{(-1)^i} \times f(g_1, \ldots, g_n)^{(-1)^{n+1}} \cdot g_{n+1}.\end{aligned}$$

We can deduce that $\operatorname{Im} d_n \subseteq \operatorname{Ker} d_{n+1}$, see also [10, p. 54]. Then consider the n^{th} *cohomology group* $H^n(G, A), n \geq 1,$ *of* G *with coefficients in* A by defining

$$H^n(G, A) := Z^n(G, A)/B^n(G, A)$$

where $Z^n(G, A) := \operatorname{Ker} d_{n+1}$ and $B^n(G, A) := \operatorname{Im} d_n$.

The second cohomology group $H^2(G, \mathbb{C}^*)$ of G with coefficients in $\mathbb{C}^*$ (the multiplicative group of complex numbers) is called the $\mathbb{C}^*$-*Schur multiplier* of G. For practical purposes it is possible to calculate the $\mathbb{C}^*$-Schur multiplier from any free presentation F/N of G: By Karpilovsky [22, p. 50, Theorem 2.4.6], for a finite group G with $G \cong F/N$, F of finite rank, we know that

$$H^2(G, \mathbb{C}^*) \cong (F' \cap N)/[F, N].$$

The n^{th} homology group $H_n(G, A)$ of G with coefficients in A can be defined as a dual concept of the n^{th} cohomology group $H^n(G, A)$. We give here a direct definition of $H_2(G, \mathbb{Z})$ which will be used later in a proof, following Karpilovsky [22, pp. 75–76]:

Let K_i $(i = 1, 2, 3)$ be the multiplicatively written abelian group freely generated by all $i-$tuples $(x_1, \ldots, x_i)$, where the $x_j \in G \backslash \{1\}$ for all $1 \leq j \leq i$. Thus

$$K_1 = \bigoplus_{x_1 \in G \backslash \{1\}} \mathbb{Z}(x_1), K_2 = \bigoplus_{x_i \in G \backslash \{1\}} \mathbb{Z}(x_1, x_2) \text{ and } K_3 = \bigoplus_{x_i \in G \backslash \{1\}} \mathbb{Z}(x_1, x_2, x_3).$$

For convenience, we put $(x_1, \ldots, x_i) = 1$ whenever some $x_j = 1$. Define a chain of homomorphisms

$$K_3 \xrightarrow{\delta_3} K_2 \xrightarrow{\delta_2} K_1$$

by the rules

$$\begin{aligned}
\delta_3(x_1, x_2, x_3) &= (x_2, x_3)(x_1x_2, x_3)^{-1}(x_1, x_2x_3)(x_1, x_2)^{-1} \\
\delta_2(x_1, x_2) &= (x_2)(x_1x_2)^{-1}(x_1)
\end{aligned}$$

on the free generators of K_2 and K_3 for $x_1, x_2, x_3 \in G \setminus \{1\}$.

It is easy to see that $\operatorname{Im}\delta_3 \subseteq \operatorname{Ker}\delta_2$. Then we define the *second homology group* $H_2(G,\mathbb{Z})$ by

$$H_2(G,\mathbb{Z}) := \operatorname{Ker}\delta_2/\operatorname{Im}\delta_3.$$

We will see in Theorem 5.3.4 that also $H_2(G,\mathbb{Z}) \cong (F' \cap N)/[F,N]$ for any free presentation $G \cong F/N$. Thus $H^2(G,\mathbb{C}^*)$ and $H_2(G,\mathbb{Z})$ are the same if G is finite which is why Ol'shanskii [31, p. 336] and many other authors refer to the second homology group $H_2(G,\mathbb{Z})$ as the *Schur multiplier* $M(G)$ of G. To show this, we need several definitions and tools.

Let F/N be a free presentation of an arbitrary group G and $\langle F,F\rangle$ the group freely generated by all pairs $\langle x,y\rangle$ with $x,y \in F$. So $\langle F,F\rangle = *_{x,y\in F}\mathbb{Z}\langle x,y\rangle$. Then define the homomorphism

$$\theta_F : \langle F,F\rangle \longrightarrow [F,F] \ (\langle x,y\rangle \mapsto [x,y]).$$

For each $x \in F$, put $\overline{x} = xN \in F/N$ and define a homomorphism θ_G by the same manner:

$$\theta_G : \langle G,G\rangle \longrightarrow [G,G] \ (\langle \overline{x},\overline{y}\rangle \mapsto [\overline{x},\overline{y}]).$$

Furthermore, we define the canonical projection

$$\overline{\pi} : \langle F,F\rangle \longrightarrow \langle G,G\rangle \ (\langle x,y\rangle \mapsto \langle \overline{x},\overline{y}\rangle).$$

Write $[w]$ for the image of $w \in \langle G,G\rangle$ in $[G,G]$, and set $C(G) := \operatorname{Ker}\theta_G$.

Each element in $C(G)$ is called a *commutator relation* of G. This relation is *universal* if it is in

$$B(G) = \overline{\pi}(\operatorname{Ker}\theta_F).$$

We have that $B(G) \trianglelefteq C(G)$ and put $H(G) := C(G)/B(G)$. Hence, $H(G)$ is the group of all commutator relations of G taken modulo universal commutator relations. This group $H(G)$ does not depend on the choice of the free presentation of G. Moreover, it is obvious and needs no proof that $H(F) = 1$ for any free group F, see e.g. [22, p. 71, Theorem 2.6.4].

If $\phi : G_1 \longrightarrow G_2$ is a group homomorphism, then we define the homomorphism

$$\phi^* : \langle G_1, G_1 \rangle \longrightarrow \langle G_2, G_2 \rangle \ (\langle x, y \rangle \mapsto \langle \phi(x), \phi(y) \rangle).$$

As ϕ^* carries $C(G_1)$ into $C(G_2)$ and $B(G_1)$ into $B(G_2)$, it induces a homomorphism

$$\phi_* : \langle G_1, G_1 \rangle / B(G_1) \longrightarrow \langle G_2, G_2 \rangle / B(G_2) \ (\langle x, y \rangle B(G_1) \mapsto \langle \phi(x), \phi(y) \rangle B(G_2))$$

which restricts to a homomorphism

$$\phi_* : H(G_1) \longrightarrow H(G_2).$$

A major step for the main result Theorem 5.3.4 is the following

Theorem 5.3.1. *Let G be an arbitrary group and $G = F/N$ a free presentation of G. Then $H(G) \cong (F' \cap N)/[F, N]$.*

Proof. For each $x \in F$, put $\overline{x} := xN \in F/N$ and $\tilde{x} := x[F, N] \in F/[F, N]$. Define

$$f : F/[F, N] \longrightarrow F/N \ (\tilde{x} \mapsto \overline{x}).$$

Since $[F, N] \subseteq N$, we have that f is a well-defined homomorphism. Furthermore, f is an epimorphism and the following sequence

$$1 \longrightarrow N/[F, N] \longrightarrow F/[F, N] \xrightarrow{f} G \longrightarrow 1$$

is exact. Thus, for any $\overline{x} \in G$ we may fix some $\tilde{x} \in F/[F, N]$ such that $f(\tilde{x}) = \overline{x}$ (e.g. by choosing a suitable pre-image $x \in F$), so we can define a homomorphism

$$\rho : \langle G, G \rangle \longrightarrow F/[F, N] \ (\langle \overline{x}, \overline{y} \rangle \mapsto [\tilde{x}, \tilde{y}]).$$

Observe that this definition of the homomorphism ρ does actually not depend on the specific choice of the pre-image $\tilde{x}$ of $\overline{x}$. Furthermore, we can verify that ρ maps $C(G)$ onto $N/[F,N] \cap [F/[F,N], F/[F,N]] = (N \cap F')/[F,N]$ and $B(G)$ onto 1. Thus, ρ induces an epimorphism

$$\psi : H(G) \twoheadrightarrow (F' \cap N)/[F,N]$$

and it is easy to check that ψ extends to an exact sequence

$$H(F/[F,N]) \xrightarrow{f_*} H(G) \xrightarrow{\psi} (F' \cap N)/[F,N] \longrightarrow 1.$$

We will show that $f_* = 0$ (modifying arguments in [22, pp. 72–73]). This will give us that $\{1_{H(G)}\} = \operatorname{Im} f_* = \operatorname{Ker} \psi$. Hence, ψ is an isomorphism and we then can conclude that $H(G) \cong (F' \cap N)/[F,N]$.

Let $\tilde{w} \in C(F/[F,N]) \subseteq \langle F/[F,N], F/[F,N] \rangle$. Then $\tilde{w} = \langle \tilde{x}_1, \tilde{y}_1 \rangle \ldots \langle \tilde{x}_k, \tilde{y}_k \rangle$ where $\tilde{x}_i, \tilde{y}_i \in F/[F,N]$ such that $[\tilde{w}] = [\tilde{x}_1, \tilde{y}_1] \ldots [\tilde{x}_k, \tilde{y}_k] = \tilde{1}$ in $F/[F,N]$. If λ is the canonical epimorphism from F to $F/[F,N]$, then choose $x_i, y_i \in F$ such that $\lambda(x_i) = \tilde{x}_i$ and $\lambda(y_i) = \tilde{y}_i$ for all $i = 1, \ldots, k$. We have $w := \langle x_1, y_1 \rangle \ldots \langle x_k, y_k \rangle \in \langle F, F \rangle$, $\lambda^*(w) = \tilde{w}$, and $[w] \in [F,N]$ because $[w][F,N] = \lambda([w]) = [\tilde{w}] = [F,N]$. So $[w]$ can in particular be written as $[w] = [s_1, r_1] \ldots [s_l, r_l]$ for some $s_i \in F$ and $r_i \in N$ (using the formula $[f,n]^{-1} = [n^{-1}fn, n]$ for all $f \in F, n \in N$). Since $\langle x_1, y_1 \rangle \ldots \langle x_k, y_k \rangle$ and $\langle s_1, r_1 \rangle \ldots \langle s_l, r_l \rangle$ have the same image under θ_F in $[F,F]$, they are in the same coset modulo $\operatorname{Ker} \theta_F = C(F)$. But F is free, this gives us $C(F) = B(F)$ and $\langle x_1, y_1 \rangle \ldots \langle x_k, y_k \rangle B(F) = \langle s_1, r_1 \rangle \ldots \langle s_l, r_l \rangle B(F)$. Then

$$wB(F) = \langle x_1, y_1 \rangle \ldots \langle x_k, y_k \rangle B(F) = \langle s_1, r_1 \rangle \ldots \langle s_l, r_l \rangle B(F)$$

and

$$\begin{aligned}
\tilde{w}B(F/[F,N]) &= \lambda^*(w)B(F/[F,N]) \\
&= \lambda_*(wB(F)) \\
&= \lambda_*(\langle s_1, r_1\rangle \dots \langle s_l, r_l\rangle B(F)) \\
&= \lambda^*(\langle s_1, r_1\rangle \dots \langle s_l, r_l\rangle)B(F/[F,N]) \\
&= \langle \lambda(s_1), \lambda(r_1)\rangle \dots \langle \lambda(s_l), \lambda(r_l)\rangle B(F/[F,N]).
\end{aligned}$$

Hence,

$$\begin{aligned}
f_*(\tilde{w}B(F/[F,N])) &= f_*(\langle \lambda(s_1), \lambda(r_1)\rangle \dots \langle \lambda(s_l), \lambda(r_l)\rangle B(F/[F,N])) \\
&= f^*(\langle \lambda(s_1), \lambda(r_1)\rangle \dots \langle \lambda(s_l), \lambda(r_l)\rangle)B(G) \\
&= \langle (f\lambda)(s_1), (f\lambda)(r_1)\rangle \dots \langle (f\lambda)(s_l), (f\lambda)(r_l)\rangle B(G).
\end{aligned}$$

Since $\lambda(r_i) \in N/[F,N] = \operatorname{Ker} f$, we have that $(f\lambda)(r_i) = \overline{1}$ for all $i = 1, \dots, l$. Therefore, we can conclude that

$$f_*(\tilde{w}B(F/[F,N])) = \langle (f\lambda)(s_1), \overline{1}\rangle \dots \langle (f\lambda)(s_l), \overline{1}\rangle B(G) = B(G)$$

because $\langle (f\lambda)(s_i), \overline{1}\rangle = \langle \overline{s_i}, \overline{1}\rangle$ for all $i = 1, \dots, l$ and $[\overline{x}, \overline{1}] = \overline{1}$ for all $x \in F$. This completes the proof. □

A *transversal* B for a subgroup S of F is a subset of F consisting of exactly one element from every right coset of S in F. Let $F = F(X)$ be freely generated by X. We say that B is a *Schreier transversal* if any reduced word $b = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n} \in B$ implies that $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_k^{\varepsilon_k} \in B$ for all its initial segments, $1 \leq k \leq n$.

We write $\overline{Sb}$ for a fixed element of the right coset Sb (e.g. $\overline{Sb}$ may denote the representative of Sb in a transversal B) and set $\overline{S} = 1$. For each $x \in X$, we put

$$t_{Sb,x} := (\overline{Sb})x(\overline{Sbx})^{-1}.$$

The famous Schreier transversal in Theorem 5.3.2 will be applied later. It can be used, among other things, to show that all subgroups of a free group are free.

Theorem 5.3.2. *Let $F = F(X)$ be a group freely generated by the set X, and let S be a subgroup of F. Then the following holds:*

(i) There exists a Schreier transversal for S in F.

(ii) If $\{\overline{Sb} \mid b \in F\}$ is a transversal for S in F, then S is generated by the set $\{t_{Sb,x} \mid b \in F, x \in X\}$.

(iii) If $\{\overline{Sb} \mid b \in F\}$ is a Schreier transversal for S in F, then S is freely generated by the set $\{t_{Sb,x} \mid t_{Sb,x} \neq 1, b \in F, x \in X\}$.

Proof. We will elaborate the proof given in [22, pp. 76–78, Theorem 2.7.1]

(i) We first define the *length* of a coset Sb as the minimum of the lengths of the elements in Sb. We will prove by induction on this length that there is a representative $\overline{Sb} \in Sb$ such that all its initial segments are representatives of cosets of shorter length. This is clear for the coset S of length 0 as $\overline{S} = 1$ by definition. By induction we may assume that Sz has length $n+1$ and let $ux^{\varepsilon} \in Sz$ where $\varepsilon = \pm 1$, $|\, u \,| = n$ and $|\, ux^{\varepsilon} \,| = n+1$. Then the coset Su has length n because of the minimality of the length $n+1$ of Sz. Thus the representative $\overline{Su} \in Su$ has already been chosen such that every initial segment of $\overline{Su}$ is also a representative. Therefore, $\overline{Sz} := (\overline{Su})x^{\varepsilon}$ is a representative of $S(\overline{Su})x^{\varepsilon} = Sux^{\varepsilon} = Sz$, as desired.

(ii) For each $x \in X$, both $(\overline{Sb})x$ and $\overline{Sbx}$ are in Sbx, i.e., $S(\overline{Sb})x = S\overline{Sbx}$. Then $t_{Sb,x} = (\overline{Sb})x(\overline{Sbx})^{-1} \in S$. To each $s \in S$ assign a free generator $y(s)$ and let Y be the free group generated by $\{y(t_{Sb,x}) \mid b \in F, x \in X\}$. Define a homomorphism

$$\alpha : Y \to S \ (y(t_{Sb,x}) \to t_{Sb,x})$$

We will show that there exists a homomorphism $\beta : S \to Y$ with $\alpha\beta = \mathrm{id}_S$. This will force α to be surjective and S is generated by $\{t_{Sb,x} \mid b \in F, x \in X\}$.

For each coset Sb, we define a map

$$\gamma^{Sb} : F \to Y \ (u \mapsto u^{Sb})$$

as follows: Set $1^{Sb} := 1$, $x^{Sb} := y(t_{Sb,x})$ and $(x^{-1})^{Sb} := y(t_{Sbx^{-1},x})^{-1}$. We extend this definition recursively to all of F by induction on $|\,u\,|$, where u is a reduced word. If $|\,u\,| = m + 1$, i.e., $u = x^{\varepsilon}v$ with $\varepsilon = \pm 1$ and $|\,v\,| = m$, define

$$u^{Sb} = (x^{\varepsilon}v)^{Sb} := (x^{\varepsilon})^{Sb}v^{Sbx^{\varepsilon}}.$$

We can verify further (by induction on $|\,u\,|$) that for all $u, v \in F$

$$(uv)^{Sb} = u^{Sb}v^{Sbu}$$
$$\alpha(u^{Sb}) = (\overline{Sb})u(\overline{Sbu})^{-1}.$$

Next, in order to get β, we restrict γ^S to S. That is,

$$\beta : S \to Y \ (u \mapsto u^S).$$

We have that also β is a homomorphism because

$$\beta(uv) = (uv)^S = u^S v^{Su} = u^S v^S = \beta(u)\beta(v)$$

for all $u, v \in S$. Moreover, for any $u \in S$

$$\alpha\beta(u) = \alpha(u^S) = \overline{S}u(\overline{Su})^{-1} = u,$$

since $\overline{Su} = \overline{S} = 1$ by definition. This completes the proof.

(iii) Let $B = \{\overline{Sb} \mid b \in F\}$ be a Schreier transversal for S in F and define a homomorphism $\phi : Y \to Y$ by setting $\phi = \beta\alpha$ where α and β are defined as in (ii). Clearly, $\phi^2 = \phi$. We will show that α restricts to an isomorphism from $\langle y(t_{Sb,x}) | t_{Sb,x} \neq 1\rangle$ to $\langle t_{Sb,x} | t_{Sb,x} \neq 1\rangle$ which generates S.

Let $N := \langle b^S | b \in B\rangle^Y$ be defined as a normal closure in Y. Furthermore, set $K := \langle y(1)\rangle^Y$ for $S \neq F$ and $K := 1$ for $S = F$. We claim that $N = K$. For $S = F$ this is trivial, as $B = \{1\}$ and $N = \langle 1^S\rangle^Y = 1 = K$. Thus, we may assume $S \neq F$. First of all, we will show that $N \subseteq K$. For this it will be sufficient to show by induction on the length of b that b^S $(b \in B)$ is a power of $y(1)$:

If $|\,b\,| = 0$, then $b = 1$ and $b^S = 1 = y(1)^0 \in K$. Inductively, let $b = vx^\varepsilon$ where $\varepsilon = \pm 1$, $|\,b\,| = n+1$ and $|\,v\,| = n$. Then, v is also in B and, by induction hypothesis, v^S is a power of $y(1)$. We have $b^S = (vx^\varepsilon)^S = v^S(x^\varepsilon)^{Sv}$ and we must distinguish the two cases $\varepsilon = 1$ and $\varepsilon = -1$:

If $\varepsilon = 1$, then $b = vx$ and $\overline{Sb} = b = vx = \overline{Svx}$. Hence $x^{Sv} = y(t_{Sv,x})$ with $t_{Sv,x} = (\overline{Sv})x(\overline{Svx})^{-1} = vxb^{-1} = 1$. If $\varepsilon = -1$, then $b = vx^{-1}$ and $\overline{Sbx} = \overline{Sv} = v$. Hence, $(x^{-1})^{Sv} = y(t_{Svx^{-1},x})^{-1} = y(t_{Sb,x})^{-1}$ with $t_{Sb,x} = (\overline{Sb})x(\overline{Sbx})^{-1} = bxv^{-1} = 1$. Thus, also $(x^\varepsilon)^{Sv}$ is a power of $y(1)$.

For $K \subseteq N$ observe that either $x \in B$ or $x^{-1} \in B$ for some $x \in X$. Thus, according to the above calculations, either $y(1) = x^S \in N$ or $y(1)^{-1} = (x^{-1})^S \in N$. This gives us that $y(1) \in N$.

Next, let $M := \langle y(t_{Sb,x})^{-1}\phi(y(t_{Sb,x})) \mid b \in F, x \in X\rangle^Y$. We will show that $M = N = K$. We first calculate the image of an arbitrary $y(t_{Sb,x})$ under ϕ: Setting $v := \overline{Sb} \in B$ and $u := \overline{Sbx} \in B$, we obtain

$$\phi(y(t_{Sb,x})) = \beta((\overline{Sb})x(\overline{Sbx})^{-1}) = (vxu^{-1})^S = v^S x^{Sv}(u^{-1})^{Svx} = v^S y(t_{Sv,x})(u^{-1})^{Su}.$$

Since $1 = (u^{-1}u)^{Su} = (u^{-1})^{Su}u^S$ and $Sb = Sv$, we have that

$$\phi(y(t_{Sb,x})) = v^S y(t_{Sb,x})(u^S)^{-1}.$$

Because $y(t_{Sb,x})^{-1}v^S y(t_{Sb,x}) \in N \trianglelefteq Y$ and

$$y(t_{Sb,x})^{-1}\phi(y(t_{Sb,x})) = (y(t_{Sb,x})^{-1}v^S y(t_{Sb,x}))(u^S)^{-1} \in N,$$

we obtain that $M \subseteq N$. If $S = F$, then $M \subseteq N = K = 1$ implies $M = N = K = 1$. If $S \neq F$, then observe $\phi(y(1)) = \beta(1) = 1^S = 1$ and $y(1)^{-1} = y(1)^{-1}\phi(y(1)) \in M$. This gives us that $y(1) \in M$, thus $K \subseteq M \subseteq N = K$ and $M = N = K$.

Let $M' := \langle y^{-1}\phi(y) | y \in Y\rangle^Y$ be another normal subgroup of Y. Consequently, $M \subseteq M'$. For any $a^{-1}\phi(a), b^{-1}\phi(b) \in M$, we have

$$(ab)^{-1}\phi(ab) = b^{-1}a^{-1}\phi(a)\phi(b) = [b^{-1}(a^{-1}\phi(a))b]b^{-1}\phi(b) \in M,$$
$$(a^{-1})^{-1}\phi(a^{-1}) = (a^{-1})^{-1}\phi(a)^{-1} = (\phi(a)a^{-1})^{-1} = [a(a^{-1}\phi(a))a^{-1}]^{-1} \in M.$$

By using these two properties and induction on the length of y, we can show that $y^{-1}\phi(y) \in M$ for all $y \in Y$. Thus $M = M'$. Moreover, we can conclude that $M' = \operatorname{Ker}\phi$, because $\phi(y^{-1}\phi(y)) = \phi(y^{-1})\phi(y) = 1$ for all $y \in Y$ and $y^{-1} = y^{-1}\phi(y) \in M'$ for all $y \in \operatorname{Ker}\phi$.

Now, we see that $K = \operatorname{Ker}\phi$. Thus $\operatorname{Ker}\phi = \langle y(1)\rangle^Y$ for $S \neq F$ and $\operatorname{Ker}\phi = 1$ for $S = F$. Then $\phi \restriction \langle y(t_{Sb,x}) \mid t_{Sb,x} \neq 1\rangle$, the restriction of ϕ to $\langle y(t_{Sb,x}) \mid t_{Sb,x} \neq 1\rangle$, is injective on $\langle y(t_{Sb,x}) \mid t_{Sb,x} \neq 1\rangle$. Since $\phi = \beta\alpha$, we obtain that also α is injective on $\langle y(t_{Sb,x}) \mid t_{Sb,x} \neq 1\rangle$. By (ii), we finally have that α is an isomorphism from $\langle y(t_{Sb,x}) \mid t_{Sb,x} \neq 1\rangle$ to $\langle t_{Sb,x} \mid t_{Sb,x} \neq 1\rangle = S$, as desired. $\square$

Let G be a group and let F be a group freely generated by the set $X = \{x_g \mid g \in G, g \neq 1\}$. We set $x_1 = 1$ and consider the epimorphism

$$\pi : F \to G \ (x_g \mapsto g).$$

We put $N := \operatorname{Ker} \pi$ and refer to the presentation $G = F/N$ as the *standard free presentation* of G.

We get the following useful

Corollary 5.3.3. *The group N defined above is freely generated by the elements $j_{p,q} := x_p x_q x_{pq}^{-1}$ for $p, q \in G$ and $p \neq 1 \neq q$.*

Proof. We will elaborate the proof given in [22, p. 78, Theorem 2.7.2].

Set $T := X \cup \{x_1\}$. We claim that T is a Schreier transversal for N in F. First, we will show that T is a transversal. As $Nb = Nx_{\pi(b)}$ for all $b \in F$, the set T does contain representatives of each coset of N in F. Assume now that $Nx_p = Nx_q$. Thus $x_p x_q^{-1} \in N = \operatorname{Ker} \pi$ and then $1 = \pi(x_p x_q^{-1}) = pq^{-1}$. This shows that $p = q$, so $x_p = x_q$. It is obvious that T is closed under taking initial segments as is required for a Schreier transversal. Applying Theorem 5.3.2(iii) and the fact that $Nx_p x_q = Nx_{pq}$ (as $\pi(x_p x_q x_{pq}^{-1}) = 1$), we can conclude that N is freely generated by the non-trivial elements of the form $\overline{Nx_p} x_q (\overline{Nx_p x_q})^{-1} = x_p x_q x_{pq}^{-1}$, where $x_p x_q x_{pq}^{-1} \neq 1$ if and only if $p \neq 1 \neq q$. $\square$

We now get the main theorem of this section.

Theorem 5.3.4. *Let G be an arbitrary group and $G = F/N$ a free presentation of G. Then $(F' \cap N)/[F, N] \cong H_2(G, \mathbb{Z})$.*

Proof. We revise arguments in [22, pp. 78–79, Theorem 2.7.3] for this proof. By Theorem 5.3.1, we know that the group $(F' \cap N)/[F, N] \cong H(G)$ is independent of the free presentation of G. Thus we may assume that $G = F/N$ is the standard free presentation. By Corollary 5.3.3, we have that N is freely generated by $j_{p,q}$ for $p, q \in G$ and $p \neq 1 \neq q$. Then $N/[N, N]$ is a free abelian group with a basis of the form $\{j_{p,q}[N, N] \mid p, q \in G, p \neq 1 \neq q\}$. Define a homomorphism of free abelian groups

$$\sigma : K_2 \longrightarrow N/[N, N] \ ((p, q) \mapsto j_{p,q}[N, N]).$$

We obtain that σ is an isomorphism, since it is defined between bases. Next, we claim that $\sigma(\mathrm{Ker}\, \delta_2) = (F' \cap N)/[N, N]$ and $\sigma(\mathrm{Im}\, \delta_3) = [F, N]/[N, N]$.

We have that

$$j_{q,r} j_{pq,r}^{-1} j_{p,qr} j_{p,q}^{-1} = j_{q,r}(x_{pq} x_r x_{pqr}^{-1})^{-1}(x_p x_{qr} x_{pqr}^{-1})(x_p x_q x_{pq}^{-1})^{-1} = j_{q,r} x_p j_{q,r}^{-1} x_p^{-1}.$$

Then

$$\sigma(\delta_3(p, q, r)) = j_{q,r} j_{pq,r}^{-1} j_{p,qr} j_{p,q}^{-1}[N, N] = [x_p, j_{q,r}]^{-1}[N, N] \in [F, N]/[N, N]$$

for all $(p, q, r) \in K_3$, and $\sigma(\mathrm{Im}\, \delta_3) \subseteq [F, N]/[N, N]$ follows. By $[x_p, j_{q,r}][N, N] \in \sigma(\mathrm{Im}\, \delta_3)$ and the identities

$$[x_p, n^{-1}][N, N] = x_p n^{-1} x_p^{-1} n[N, N] = n^{-1}[x_p, n]^{-1} n[N, N] = [x_p, n]^{-1}[N, N],$$

$$\begin{aligned} [x_p, n_1 n_2][N, N] &= x_p n_1 n_2 x_p^{-1} n_2^{-1} n_1^{-1}[N, N] \\ &= [x_p, n_1] n_1 x_p n_2 x_p^{-1} n_2^{-1} n_1^{-1}[N, N] \\ &= [x_p, n_1] n_1 [x_p, n_2] n_1^{-1}[N, N] \\ &= [x_p, n_1][x_p, n_2][N, N] \end{aligned}$$

for all $n, n_1, n_2 \in N$ follows recursively that also $[x_p, n][N, N] \in \sigma(\operatorname{Im}\delta_3)$ for all $n \in N$. Furthermore, with Corollary 5.3.3 any $f \in F$ can be written as $f = n'x_p$ for some $n' \in N$ and $p \in G$ and

$$\begin{aligned}[f, n][N, N] &= [n'x_p, n][N, N] \\ &= n'x_p n x_p^{-1}(n')^{-1}n^{-1}[N, N] \\ &= n'[x_p, n]n(n')^{-1}n^{-1}[N, N] \\ &= [x_p, n][N, N] \in \sigma(\operatorname{Im}\delta_3)\end{aligned}$$

for all $f \in F, n \in N$. This proves the converse $[F, N]/[N, N] \subseteq \sigma(\operatorname{Im}\delta_3)$.

Next, let $a \in \operatorname{Ker}\delta_2$. Then a can be written as a finite product of generators of K_2, say $a = \prod_{p,q\in G\setminus\{1\}}(p, q)^{n_{p,q}}, n_{p,q} \in \mathbb{Z}$.

Note that

$$1 = \delta_2(a) = \prod_{p,q\in G\setminus\{1\}} ((q)(pq)^{-1}(p))^{n_{p,q}} \tag{5.4}$$

if and only if the power of each generator in the product is zero.

We have that $\sigma(a) = \sigma(\prod_{p,q\in G\setminus\{1\}}(p, q)^{n_{p,q}}) = \prod_{p,q\in G\setminus\{1\}} j_{p,q}^{n_{p,q}}[N, N] \in N/[N, N]$. Set $b := \prod_{p,q\in G\setminus\{1\}} j_{p,q}^{n_{p,q}}$. Since $b \in F$, we have $b[F, F] \in F/[F, F]$. But

$$b[F, F] = \prod_{p,q\in G\setminus\{1\}} j_{p,q}^{n_{p,q}}[F, F] = \prod_{p,q\in G\setminus\{1\}} ((x_p)(x_q)(x_{pq})^{-1})^{n_{p,q}}[F, F],$$

which is a product of the generators of the free abelian group $F/[F, F]$ of the same form as in (5.4). This forces $b[F, F] = [F, F]$, i.e., $b \in [F, F]$. Hence, we obtain that $\sigma(a) = b[N, N] \in (F' \cap N)/[N, N]$. To show that $(F' \cap N)/[N, N] \subseteq \sigma(\operatorname{Ker}\delta_2)$, let $b[N, N] \in (F' \cap N)/[N, N]$. We have that b is of the form $b = \prod_{p,q\in G\setminus\{1\}} j_{p,q}^{n_{p,q}}$ because $b \in N$. In addition, $1 = b[F, F] \in F/[F, F]$ since $b \in F'$. We now can reverse the above arguments to show $\sigma(a) \in b[N, N]$ and

$\delta_2(a) = 1$ for $a := \prod_{p,q\in G\setminus\{1\}}(p,q)^{n_{p,q}}$. We now can conclude that

$$H_2(G,\mathbb{Z}) = \operatorname{Ker}\delta_2/\operatorname{Im}\delta_3 \cong \sigma(\operatorname{Ker}\delta_2)/\sigma(\operatorname{Im}\delta_3) = (F'\cap N)/[F,N].$$

This completes the proof. □

5.4 Burnside varieties which are neither cellular closed nor finitely based

Here we will apply a characterization of Schur multipliers from [22] to obtain the existence of countably many varieties as in the title. Whether or not there are $2^{\aleph_0}$ many such varieties still remains a question to be answered.

By Theorem 5.2.3, we know that the system of group laws (5.2) describes a variety which is not finitely based. We will show that this variety is not closed under cellular covers either by establishing a group in the variety such that its cellular cover is outside the variety. To do so, we will modify the arguments given in [13, pp. 328–329].

We consider a prime $p > 10^{75}$ and the Burnside group $\mathcal{B}$ of the Burnside variety $\mathfrak{B}_p$ of exponent p constructed in Section 3.3. We recall some necessary properties of $\mathcal{B}$ from Section 3.3 and [31, p. 276, Lemma 25.12]:

(i) $\mathcal{B}$ is infinite.

(ii) Every non-trivial proper subgroup of $\mathcal{B}$ is cyclic of order p.

(iii) $\mathcal{B}$ is generated by two elements.

(iv) If $x, y \in \mathcal{B}, n \in \mathbb{Z}$ such that $x^n \neq 1$ and $x^n y = yx^n$, then $xy = yx$.

Furthermore, we obtain that $\mathcal{B}$ has one more important property.

Lemma 5.4.1. *If the group B satisfies Conditions (i),(ii), (iii) and (iv), then B is simple.*

Proof. First, we will show that $\mathfrak{z}B = 1$: With Conditions (i),(ii) and (iii) the group B is non-abelian, thus $\mathfrak{z}B \neq B$. Hence, there is an element $x \in B \setminus \mathfrak{z}B$ and $\mathfrak{z}B$ is cyclic of finite order. Because $\mathfrak{z}B$ is normal and x is of order p, we have that $\langle x, \mathfrak{z}B\rangle = \langle x\rangle\mathfrak{z}B$ is a proper finite subgroup of B, hence $\langle x, \mathfrak{z}B\rangle = \langle y\rangle$ for some $y \in B$. Suppose on the contrary that $\mathfrak{z}B \neq 1$. Then there exists $1 \neq z \in \mathfrak{z}B$, and $z = y^n$ for some integer n. Consequently, $y^n g = zg = gz = gy^n$ for all $g \in B$. By Condition (iv), we obtain that $yg = gy$ for all $g \in B$, i.e., $y \in \mathfrak{z}B$. Thus, $x \in \langle y\rangle \subseteq \mathfrak{z}B$, which is a contradiction.

To show that B is simple, suppose that $N \neq B$ is a normal subgroup of B, say $N = \langle a\rangle$ for some $a \in B$. Let g be any element in B. Then, by the same argument, $\langle g, N\rangle = \langle g\rangle N = \langle b\rangle$ for some $b \in B$, and hence $g = b^r$ and $a = b^s$ for some integers r, s. Thus $ga = b^r b^s = b^s b^r = ag$. This shows that $a \in \mathfrak{z}B = 1$. Hence, we have $N = \langle a\rangle = 1$. Therefore, only the trivial subgroup of B and B itself are normal in B. $\square$

Let F/N be a free presentation of our special Burnside group $\mathcal{B}$ where $F = F(\{x_1, x_2\})$ and $N = \mathcal{R}^F$ is the normal closure of the set of relators $\mathcal{R} = \bigcup_{i<\omega}\mathcal{R}_i = \bigcup_{0<i<\omega}\mathcal{S}_i$. Then, by Theorem 3.3.15, $N/[F,N]$ is free abelian. Since the Schur multiplier $M(\mathcal{B}) = (F' \cap N)/[F,N]$ is a subgroup of $N/[F,N]$, $M(\mathcal{B})$ is free abelian as well, see Theorem 2.1.6. Hence, $M(\mathcal{B})$ is *torsion-free*, i.e., apart from the identity every element is of infinite order. The next lemma shows that $M(\mathcal{B})$ is actually not the trivial group, making this a valuable observation.

Lemma 5.4.2. *The Schur multiplier $M(\mathcal{B})$ is free abelian of rank > 0.*

Proof. We will compare the construction of $\mathcal{B}$ to that of $B(\{x_1, x_2\}, p)$ from Section 3.3. For this purpose (while avoiding confusion) let $F/\overline{N}$ denote the respective free presentation of $B(\{x_1, x_2\}, p)$ where $F = F(\{x_1, x_2\})$ and $\overline{N} = \overline{\mathcal{R}}^F$ is the normal closure of the set of relators $\overline{\mathcal{R}} = \bigcup_{i<\omega} \overline{\mathcal{R}_i} = \bigcup_{0<i<\omega} \overline{\mathcal{S}_i}$.

We claim that $|\mathcal{R}| \geq 3$: First observe that without loss of generality we may set $\overline{\mathcal{S}_1} = \{x_1^p, x_2^p\} \subseteq \mathcal{S}_1$, there may however exist additional relators of type 2 in $\mathcal{S}_1$. If $|\mathcal{S}_1| \geq 3$, then we are done. Thus let us assume $\mathcal{S}_1 = \overline{\mathcal{S}_1} = \{x_1^p, x_2^p\}$. By [31, p. 214, Theorem 19.3], the set $\overline{\mathcal{R}}$ is infinite. Hence, there exists some minimal $j \geq 2$ with $\overline{\mathcal{S}_j} \neq \emptyset$. But then we may set without loss of generality $\mathcal{S}_i = \overline{\mathcal{S}_i}$ for all $1 \leq i < j$ and $\overline{\mathcal{S}_j} \subseteq \mathcal{S}_j$, and $|\mathcal{R}_j| \geq 3$ follows. By Theorem 3.3.15 and the claim, $N/[F, N]$ is free abelian of rank $|\mathcal{R}| \geq 3$, while

$$(N/[F, N])/M(\mathcal{B}) \cong N/(F' \cap N) \cong F'N/F' \subseteq F/F'$$

is free abelian of rank ≤ 2. Thus $M(\mathcal{B})$ is a direct summand of $N/[F, N]$ and free abelian of rank ≥ 1. $\square$

Remark 5.4.3. *By similar means it is possible to show that the Schur multiplier $M(B(\{x_1, x_2\}, p))$ is free abelian of countable rank, cf. [31, p. 336, Corollary 31.2]. Observe also that unlike the free Burnside group $B(\{x_1, x_2\}, p)$ the structure of the group $\mathcal{B}$ may be dependent on the specific choice of the set $\mathcal{R}$ of defining relators. Thus, more accurately, Lemma 5.4.2 shows that the construction of $\mathcal{B}$ may be modified to guarantee $M(\mathcal{B})$ to be non-trivial.*

For the next step, we need some more tools: A group G is said to be *perfect* if it is equal to its own commutator subgroup, i.e., $G = G' = [G, G]$, see e.g. [32, p. 157]. An epimorphism $f : G \longrightarrow Q$ is called a *perfect cover* of Q if G is perfect and $\operatorname{Ker} f \subseteq \mathfrak{z}G$, following [2, p. 113].

Lemma 5.4.4. *The group $\mathcal{B} = F/N$ has a central extension*

$$1 \longrightarrow K \longrightarrow A \xrightarrow{e} \mathcal{B} \longrightarrow 1,$$

where $K := (F' \cap N)/[F,N]$, $A := F'/[F,N]$, and

$$e : F'/[F,N] \longrightarrow F/N \ (f[F,N] \mapsto fN)$$

is a perfect cover.

Remark 5.4.5. *This central extension is known as the universal perfect cover of $\mathcal{B}$ and has a number of important properties, see [2, p. 115, Remark 5.4] and [22, p. 94, Theorem 2.10.3]. In a similar fashion, a universal perfect cover is possible for any perfect group. Observe here that $\mathcal{B}$ is clearly perfect as it is a simple and non-abelian group.*

Proof. Let K, A and e be defined as in the statement of the lemma. Clearly, $e : A \longrightarrow \mathcal{B} = F/N$ is well-defined as $[F,N] \subseteq N$. It is obvious that e is a homomorphism and that $\operatorname{Ker} e = (F' \cap N)/[F,N] = K \subseteq \mathfrak{z}A$. In order to show that e is also surjective, we use the fact that $\mathcal{B} = F/N$ is perfect, and let y be any element in $F/N = [F/N, F/N] = F'N/N$. Then $y = fN$ for some $f \in F'$, and hence we easily choose $f[F,N] \in F'/[F,N]$ so that its image under e is equal to y.

It remains to show that $A = F'/[F,N]$ is a perfect group: For this we must verify $F'/[F,N] = A = [A,A] = F''[F,N]/[F,N]$, i.e., that $F' = F''[F,N]$ holds. The inclusion $F''[F,N] \subseteq F'$ is clear. For the proof of $F' \subseteq F''[F,N]$, let $y_1, y_2 \in F$ be given. As $F = F'N$, we may write $y_i = f_i n_i, i = 1,2$, with $f_i \in F'$ and $n_i \in N$. We next use the commutator identities

$$[ab, c] = [a,c][[a,c],b][b,c] \quad \text{and} \quad [a, bc] = [a,c][a,b][[a,b],c]$$

to write

$$[y_1, y_2] = [f_1, y_2][[f_1, y_2], n_1][n_1, y_2] = [f_1, n_2][f_1, f_2][[f_1, f_2], n_2][[f_1, y_2], n_1][n_1, y_2].$$

Then we add the identity $1 = [f_1, f_2][f_1, f_2]^{-1}$ to the right hand side and regroup them as

$$[y_1, y_2] = [f_1, f_2]([f_1, f_2]^{-1}[f_1, n_2][f_1, f_2])[[f_1, f_2], n_2][[f_1, y_2], n_1][n_1, y_2].$$

We conclude $[y_1, y_2] \in F''[F, N]$ as $[F, N] \trianglelefteq F$ and therefore $F' \subseteq F''[F, N]$. $\square$

Next, we will show that $e : A \longrightarrow \mathcal{B}$ in Lemma 5.4.4 is a cellular cover of $\mathcal{B}$. It suffices to show that $e : A \longrightarrow \mathcal{B}$ is A-equivalent, i.e., to show that the homomorphism

$$e_* : \mathrm{Hom}(A, A) \longrightarrow \mathrm{Hom}(A, \mathcal{B}) \ (\overline{\varphi} \mapsto \overline{\varphi}e)$$

is bijective.

First, we claim that e_* is surjective: Let $\varphi \in \mathrm{Hom}(A, \mathcal{B})$. We must find $\overline{\varphi} \in \mathrm{Hom}(A, A)$ such that $\overline{\varphi}e = \varphi$. If $\varphi = 0$, then we choose $\overline{\varphi} = 0$. So $\overline{\varphi}e = \varphi$ clearly holds. Next, we assume that $\varphi \neq 0$, then $1 \neq A\varphi \subseteq \mathcal{B}$. If $A\varphi \neq \mathcal{B}$, then, by the choice of $\mathcal{B}$, $A\varphi$ is cyclic hence abelian. Since A is perfect, we have that

$$A\varphi = [A, A]\varphi = [A\varphi, A\varphi],$$

i.e., $A\varphi$ is perfect as well. Consequently, we obtain that $A\varphi = [A\varphi, A\varphi] = 1$ which contradicts the assumption that $\varphi \neq 0$. Thus, we now have that $A\varphi = \mathcal{B}$, i.e., φ is surjective. From the proof of Lemma 5.4.1, we know that $\mathfrak{z}\mathcal{B} = 1$. Moreover, we can see that $(\mathfrak{z}A)\varphi \subseteq \mathfrak{z}\mathcal{B} = 1$ because φ is surjective. From the above central extension, we know that $K \subseteq \mathfrak{z}A$ and $A/K \cong \mathcal{B}$. In addition, the fact that A is perfect implies that A is non-abelian, hence $\mathfrak{z}A \neq A$. Thus, $\mathfrak{z}A/K \subseteq A/K$ is a proper normal subgroup of the simple group A/K. This forces $K = \mathfrak{z}A$, $K \subseteq \operatorname{Ker}\varphi$ and $\varphi = e\beta$ follows for some $\beta \in \operatorname{Hom}(\mathcal{B}, \mathcal{B})$. Next, we claim that $\beta \in \operatorname{Aut}(\mathcal{B})$. If $\varphi = e\beta$ for some $\beta \in \operatorname{Hom}(\mathcal{B}, \mathcal{B})$, then $\operatorname{Ker}\beta = 1$ because $\operatorname{Ker}\beta \trianglelefteq \mathcal{B}$, $\mathcal{B}$ is simple and $\varphi \neq 0$, i.e., β is injective. We can see that β is also surjective by using that φ is surjective. Thus, we finally have that $\beta \in \operatorname{Aut}(\mathcal{B})$ and $\operatorname{Ker}\varphi \subseteq \operatorname{Ker} e = K$, hence $\operatorname{Ker}\varphi = K$.

We next consider the following diagram

$$\begin{tikzcd}
1 \arrow[r] & N/[F,N] \arrow[r] & F/[F,N] \arrow[r, "\lambda"] \arrow[d, "\alpha"] & \mathcal{B} \arrow[r] \arrow[d, "\beta"] & 1 \\
1 \arrow[r] & K \arrow[r] & A \arrow[r, "e"] & \mathcal{B} \arrow[r] & 1
\end{tikzcd}$$

where λ is an epimorphism defined by sending $f[F,N] \in F/[F,N]$ to $fN \in F/N$. By Lemma 2.1.11, there exists a homomorphism $\alpha : F/[F,N] \longrightarrow A$ such that the above diagram commutes, i.e., $\alpha e = \lambda\beta$. Recall that $A = F'/[F,N] \subseteq F/[F,N]$. Hence, $\alpha : F/[F,N] \longrightarrow A$ can be restricted to $\overline{\varphi} := \alpha \restriction A$. Then $\overline{\varphi} \in \operatorname{Hom}(A, A)$. But $e = \lambda \restriction A$, then we have that $\varphi = e\beta = \overline{\varphi}e$.

Next, we will show that e_* is injective: Let $\varphi_1, \varphi_2 \in \operatorname{Hom}(A, A)$ such that $\varphi_1 e = \varphi_2 e$. Define

$$\psi : A \longrightarrow A \ (a \mapsto a\varphi_1 a^{-1}\varphi_2).$$

It is easy to see that $A\psi \subseteq \operatorname{Ker} e = K$. For $a, b \in A$, we obtain that

$$(ab)\psi = (ab)\varphi_1(ab)^{-1}\varphi_2 = a\varphi_1(b\varphi_1 b^{-1}\varphi_2)a^{-1}\varphi_2 = a\varphi_1 b\psi a^{-1}\varphi_2.$$

Using that $K = \mathfrak{z}A$, since $b\psi \in K$ and $a^{-1}\varphi_2 \in A$, we then obtain

$$(ab)\psi = a\varphi_1 a^{-1}\varphi_2 b\psi = a\psi b\psi,$$

i.e., $\psi \in \operatorname{Hom}(A, K)$. Moreover, as A is perfect and $A\psi \subseteq K = \mathfrak{z}A$ is abelian, we obtain

$$A\psi = [A, A]\psi = [A\psi, A\psi] = 1.$$

As a result, $\psi = 0$ and then

$$1 = a\varphi_1 a^{-1}\varphi_2 = a\varphi_1(a\varphi_2)^{-1}$$

for all $a \in A$. This shows that $\varphi_1 = \varphi_2$. Thus e_* is injective, as desired.

Therefore, $e : A \longrightarrow \mathcal{B}$ is a cellular cover of $\mathcal{B}$. But A contains $K = M(\mathcal{B})$ which contains a copy of $\mathbb{Z}$ by Lemma 5.4.2, hence A does not satisfy any law in the system of group laws (5.2). Therefore, A cannot belong to the variety generated by the system of group laws (5.2) while $\mathcal{B}$ evidently does. This shows that this variety is not closed under taking cellular covers.

By Theorem 5.2.4, we now can establish the following

Theorem 5.4.6. *There exist countably many pairwise distinct varieties of groups which are neither cellular closed nor finitely based.*

Chapter 6

$2^{\aleph_0}$ varieties of groups not closed under cellular covers

In this chapter we will establish the existence of $2^{\aleph_0}$ varieties of groups which are not closed under cellular covers and consequences of this new result.

6.1 A survey and some modifications of Ol'shanskii's results from [29]

According to Birkhoff, every variety can be described by a suitable system of laws. As the set of possible laws has cardinality $\aleph_0$, this immediately implies that there exist not more than $2^{\aleph_0}$ varieties, and the question whether there indeed exist $2^{\aleph_0}$ pairwise distinct varieties originates from B. H. Neumann [26] in 1937. One might expect that a positive answer to this question can be given as follows: Start with a countable system of laws which is not equivalent to any finite system of laws, cf. Theorem 5.2.3. It should then be possible to construct $2^{\aleph_0}$ pairwise distinct varieties by choosing suitable subsets of this system of laws. Unfortunately, not every infinite system of laws qualifies for this approach and, in particular, the system of laws (5.2) is unsuitable as every law implies all its predecessors (replacing some variables by the identity). Thus we will follow instead the ideas of a construction presented by Ol'shanskii [29] in 1970. Note that there are also other constructions of $2^{\aleph_0}$ varieties of groups, for example by Adjan [1] and Vaughan-Lee [37].

We begin by elaborating the proof of Theorem A given in [35] in the following

Theorem 6.1.1. *Let G be a monolithic group with the monolith L containing an element of prime order q. If p is a prime different from q, then G has a faithful irreducible linear representation $\varphi : G \longrightarrow GL(M)$ over the field $\mathbb{Z}_p$.*

Proof. Let $C = \langle x \rangle$ be a cyclic group of order p. We consider the restricted wreath product $W = C \operatorname{wr} G$ of C and G, which is a semidirect product $W = B \rtimes G$ where $B = \bigoplus_{g \in G} C_g = \bigoplus_{g \in G} \langle x_g \rangle$, cf. Section 2.1. We will identify C and C_1. Note also that B can be regarded as a vector space over the field $\mathbb{Z}_p$ of dimension $|\,G\,|$.

Assume that y is the element of order q in L and put $\xi := [x, y]$. We have that $\xi = (x_1^{-1}, y^{-1})(x_1, y) = (x_1^{-1}\alpha_{y^{-1}}(x_1), 1) = x_1^{-1}x_{y^{-1}}$, thus $1 \neq \xi \in B$. Let $U := \langle \xi \rangle^W$ be the normal closure of ξ in W. It is obvious that $U \subseteq B$ since B is normal in W. Then consider the set $M' := \{U' \subseteq U \mid U' \trianglelefteq W \text{ and } \xi \notin U'\}$. Clearly, M' is not empty as the trivial group is an element of M', and M' is a partially ordered set under inclusion. Moreover, M' is closed under taking unions of chains. Thus, by Zorn's lemma, we obtain a maximal element V in

M'. We know that V is also contained in B.

If $M := U/V$, then M is a vector space over $\mathbb{Z}_p$ since $U, V \subseteq B$. By maximality of V, there is no normal subgroup V' of W such that $V \subset V' \subset U$.

Next, we consider the following linear representation of G over $\mathbb{Z}_p$:

$$\varphi : G \longrightarrow GL(M) \ (g \mapsto \overline{g})$$

where $\overline{g}$ is defined as follows

$$\overline{g} : M \longrightarrow M \ (uV \mapsto (g^{-1}ug)V).$$

It is easy to see that $\overline{g}$ is well-defined because V is normal, and $\overline{g}$ is actually an automorphism on M with inverse $\overline{g^{-1}}$. Moreover, we can see that this representation is irreducible, because any non-trivial proper subspace of M invariant under $\overline{g}$ for all $g \in G$ gives a normal subgroup of W lying between V and U.

We need to show further that φ is faithful, i.e., $\operatorname{Ker}\varphi = 1$. Suppose on the contrary that $\operatorname{Ker}\varphi \neq 1$. Since $\operatorname{Ker}\varphi$ is normal in G and L is the monolith of G, we have that $L \subseteq \operatorname{Ker}\varphi$. It follows that, for any $l \in L$ and $u \in U$, we have $\overline{l} = l\varphi = \overline{1}$ and hence $(l^{-1}ul)V = (uV)\overline{l} = uV$. So, $[u, l] \in V$ for all $l \in L$ and $u \in U$. This shows that $[U, L] \subseteq V$ and hence $[[x, y], y] \in V$. As B is abelian, also $[[x, y], x] = 1 \in V$ holds and $[x, y]V \in \mathfrak{z}N$ follows for $N := \langle xV, yV \rangle$. Furthermore, $N' = \langle [x, y]V \rangle^N \subseteq \mathfrak{z}N$. Thus we conclude that $[N, N'] = 1$ and N is a nilpotent subgroup of class 2 of the group W/V. Note that N is finite as $\langle x, y \rangle = \langle x \rangle \operatorname{wr} \langle y \rangle \subseteq W$ with x and y of finite order. Consequently, N is the cartesian product of its Sylow subgroups (see Theorem 2.1.5). Let N_p and N_q be the maximal p-subgroup and the maximal q-subgroup of N, respectively. We have $xV \in N_p$ and $yV \in N_q$, and we obtain that $[x, y]V = [xV, yV] \in N_p \cap N_q$ since both N_p and N_q are normal. But $N_p \cap N_q = 1_N = V$ because p and q are relatively prime. This gives us that $[x, y] \in V$, which contradicts the choice of V. Therefore, φ is a faithful irreducible linear representation of G over $\mathbb{Z}_p$, as desired. $\square$

We get the following consequences.

Corollary 6.1.2. *Under the assumptions of Theorem 6.1.1, G can be embedded in a monolithic group $S = M \rtimes G$ with monolith M.*

Proof. From the linear representation $\varphi : G \longrightarrow GL(M)$, we can construct a semidirect product $S = M \rtimes G$ of M and G: We define

$$\varphi^* : G \longrightarrow GL(M)\ (g \mapsto \overline{g^{-1}})$$

and $S := M \rtimes_{\varphi^*} G$, and we will temporarily write the homomorphisms $\overline{g}$ on the left hand side.

It remains to show that M is the unique minimal normal subgroup $\neq 1$ of S. Let N be any non-trivial normal subgroup of S. In order to show that $M \subseteq N$, we consider an arbitrary non-identity element (m, g) in N, where $m \in M$ and $g \in G$, since $N \subseteq S = M \rtimes_{\varphi^*} G$.

Case (i): $g \neq 1$

Since φ is faithful, we have $\overline{g^{-1}} \neq \overline{1}$. Then, there exists an element $n \in M$ such that $\overline{g^{-1}}(n) \neq n$. We have that $(m, g)^n \in N$ as N is normal in S, hence $(m, g)^n(m, g)^{-1} \in N$. By easy calculation and recalling that M is abelian, we have

$$(m, 1)^n = (n^{-1}, 1)(m, 1)(n, 1) = (n^{-1}m, 1)(n, 1) = (n^{-1}mn, 1) = (m, 1),$$

$$(1, g)^n = (n^{-1}, 1)(1, g)(n, 1) = (n^{-1}, g)(n, 1) = (n^{-1}\overline{g^{-1}}(n), g) \neq (1, g).$$

Then

$$(m,g)^n = (m,1)^n(1,g)^n \neq (m,g),$$

hence

$$(m,g)^n(m,g)^{-1} \neq 1.$$

We finally get an element of the form

$$(m',1) := (m,g)^n(m,g)^{-1} \in N$$

for some $1 \neq m' \in N$, which leads us to Case (ii).

Case (ii): $g = 1$

Since (m,g) is not the identity element of $M \rtimes_{\varphi^*} G$, we have that $m \neq 1$. By using the fact that N is normal in S, we obtain that $(m,1)^h \in N$ for all $h \in G$. But

$$\begin{aligned}(m,1)^h &= (1,h)^{-1}(m,1)(1,h)\\ &= (1,h^{-1})(m,1)(1,h)\\ &= (\overline{h}(m),h^{-1})(1,h)\\ &= (\overline{h}(m),1)\end{aligned}$$

and the group $\langle(\overline{h}(m),1) \mid h \in G\rangle \subseteq M$ is a subspace of M invariant under the group action. Thus this group is M because φ is irreducible. This shows that $M \subseteq N$ and the corollary holds immediately. $\square$

Corollary 6.1.3. *Under the assumptions of Theorem 6.1.1 and Corollary 6.1.2 follows $C_S(M) = M$ for the centralizer of the subgroup M of S.*

Proof. Using that M is abelian, we immediately get $M \subseteq C_S(M)$.

It remains to show that $C_S(M) \subseteq M$. To show this, let $s \in S \setminus M$. Then $s = mg$ for some $m \in M$ and $1 \neq g \in G$. Since φ is faithful, we have $\overline{g} \neq \overline{1}$, i.e., there exists an element $m_0 \in M$ such that $m_0 \neq m_0\overline{g} = g^{-1}m_0g$. So, $m_0g \neq gm_0$ and hence, multiplying by m on both sides and using that M is abelian, we obtain $m_0s \neq sm_0$. This shows that $s \notin C(M)$ and completes the proof. □

We denote the variety of abelian groups of exponent n by $\mathfrak{A}_n$. Now we are ready to present the details of the proof of Lemma 1 given in [29].

Lemma 6.1.4. *Assume that there exists an infinite series of finite groups T_i, $i \in \omega$, such that for each i*

a) T_i is from a fixed locally finite variety $\mathfrak{V} \subseteq \mathfrak{B}_e$,

b) T_i is monolithic, and

c) T_i is not isomorphic to any factor of T_j for $i \neq j$.

If p is a prime not dividing e, then the product $\mathfrak{A}_p\mathfrak{V}$ has $2^{\aleph_0}$ pairwise distinct subvarieties.

Proof. By Theorem 6.1.1 and Corollary 6.1.2, we have that T_i has a faithful irreducible linear representation over $\mathbb{Z}_p$ corresponding to a semidirect product $S_i = M_i \rtimes T_i$ where $M_i \in \mathfrak{A}_p$ and $S_i \in \mathfrak{A}_p\mathfrak{V}$.

We will show that any two varieties generated by distinct sets consisting of groups S_i are different. That is, we will show that

$$\mathfrak{U}_I \neq \mathfrak{U}_J \text{ for all } I \neq J \subseteq \omega \text{ where } \mathfrak{U}_I := \mathrm{var}\{S_i \mid i \in I\}. \tag{6.1}$$

Then, there are $2^{\aleph_0}$ pairwise distinct subvarieties $\mathfrak{U}_I$ of $\mathfrak{A}_p\mathfrak{V}$ corresponding to the $2^{\aleph_0}$ subsets of ω.

First, we will claim that if $S_i \notin \mathrm{var}\{S_0, \dots, S_{i-1}, S_{i+1}, \dots, S_n\}$ for any pair of non-negative integers i and n, then (6.1) holds.

Assume that $S_i \notin \mathrm{var}\{S_0, \dots, S_{i-1}, S_{i+1}, \dots, S_n\}$ for all non-negative integers i and n, and that $I \neq J \subseteq \omega$. Then, without loss of generality, there is an integer $i \in I$ but $i \notin J$. Clearly, $S_i \in \mathfrak{U}_I$ since S_i is a generator of $\mathfrak{U}_I$. We will show that $S_i \notin \mathfrak{U}_J$.

If J is finite, let n be the biggest integer in J. Then $J \subseteq \{0, 1, \dots, n\} \setminus \{i\} =: \overline{J}$. Since $J \subseteq \overline{J}$, we obtain that $\mathfrak{U}_J \subseteq \mathfrak{U}_{\overline{J}}$ (there are more generators in $\mathfrak{U}_{\overline{J}}$). By assumption, we know that $S_i \notin \mathfrak{U}_{\overline{J}}$. It follows that $S_i \notin \mathfrak{U}_J$.

If J is infinite, consider the variety $\mathfrak{U}_\omega = \mathrm{var}\{S_0, S_1, S_2, \dots\}$ generated by all groups S_i and apply Remark 2.2.8 as follows:

We have that $\mathfrak{A}_p$, the variety of abelian groups of exponent p, is locally finite, by the fundamental theorem of finitely generated abelian groups, and $\mathfrak{V}$ is locally finite by assumption. Then $\mathfrak{A}_p\mathfrak{V}$, containing all extensions of groups in $\mathfrak{A}_p$ by groups in $\mathfrak{V}$, is locally finite as well, cf. Corollary 3.2.4. Hence, its subvarieties $\mathfrak{U}_\omega$ and $\mathfrak{U}_J \subseteq \mathfrak{U}_\omega$ are locally finite. Observe that T_i, M_i and $S_i = M_i \rtimes T_i$ are finite groups. If $S_i \in \mathfrak{U}_J$, then, by Remark 2.2.8, there exists some finite $J' \subseteq J$ with $S_i \in \mathfrak{U}_{J'}$ in contradiction to the previous finite case. Hence, $S_i \notin \mathfrak{U}_J$ even when J is infinite. This completes the proof of the claim.

Now it suffices to prove that

$$S_i \notin \mathrm{var}\{S_0, \dots, S_{i-1}, S_{i+1}, \dots, S_n\} \text{ for all non-negative integers } i \text{ and } n.$$

Hence, we assume the contrary and derive a contradiction as follows:

Step (i): Suppose, without loss of generality, that $S_0 \in \mathrm{var}\{S_1, ..., S_n\}$. By Corollary 6.1.2, we know that S_0 is monolithic with monolith M_0. We apply Lemma 2.2.9 to a minimal presentation of $S_0 \neq 1$ in the locally finite variety

$\mathrm{var}\{S_1, ..., S_n\} \subseteq \mathfrak{A}_p\mathfrak{V}$ to obtain a group D with the following properties:

- The group D is a factor of one of the groups $S_1, ..., S_n$, say S_1.
- The group D is critical. Thus, by Theorem 2.2.4, D is monolithic with monolith M.
- Furthermore, M_0 in S_0 is similar to M in D, i.e., there exist isomorphisms

$$\Phi : M_0 \longrightarrow M \quad \text{and} \quad \Psi : S_0/C_{S_0}(M_0) \longrightarrow D/C_D(M)$$

such that for all $m \in M_0$ and $g \in S_0/C_{S_0}(M_0)$

$$(m^g)\Phi = (m\Phi)^{g\Psi}. \tag{6.2}$$

By Corollary 6.1.3, we obtain that $C_{S_0}(M_0) = M_0$. Then we have

$$S_0 = M_0 \rtimes T_0 \cong M_0 \rtimes S_0/M_0 = M_0 \rtimes S_0/C_{S_0}(M_0),$$

where the multiplication is given for $m_1, m_2 \in M_0$, $l_1, l_2 \in S_0/C_{S_0}(M_0)$ by

$$(m_1, l_1)(m_2, l_2) = (m_1 m_2^{l_1^{-1}}, l_1 l_2).$$

Step (ii): Similarly to $M_0 \rtimes S_0/C_{S_0}(M_0)$ we may define a semidirect product $M \rtimes D/C_D(M)$, where the multiplication is given for $m_1', m_2' \in M$, $l_1', l_2' \in D/C_D(M)$ by

$$(m_1', l_1')(m_2', l_2') = (m_1'(m_2')^{(l_1')^{-1}}, l_1' l_2').$$

We claim that there is an isomorphism between $M_0 \rtimes S_0/C_{S_0}(M_0)$ and $M \rtimes D/C_D(M)$. We begin by defining a map

$$\eta : M_0 \rtimes S_0/C_{S_0}(M_0) \longrightarrow M \rtimes D/C_D(M) \ ((m, l) \mapsto (m\Phi, l\Psi)).$$

We have that η is a homomorphism since the following holds for any $m_1, m_2 \in M_0$ and $l_1, l_2 \in S_0/C_{S_0}(M_0)$:

$$\begin{aligned} ((m_1, l_1)(m_2, l_2))\eta &= (m_1 m_2^{l_1^{-1}}, l_1 l_2)\eta \\ &= ((m_1 m_2^{l_1^{-1}})\Phi, (l_1 l_2)\Psi) \\ &= (m_1\Phi(m_2^{l_1^{-1}})\Phi, l_1\Psi l_2\Psi) \end{aligned} \tag{6.3}$$

and

$$(m_1, l_1)\eta(m_2, l_2)\eta = (m_1\Phi, l_1\Psi)(m_2\Phi, l_2\Psi) = (m_1\Phi(m_2\Phi)^{l_1^{-1}\Psi}, l_1\Psi l_2\Psi), \tag{6.4}$$

where Property (6.2) implies the equality of (6.3) and (6.4). Moreover, we can deduce that η is bijective by applying that Φ and Ψ are bijective.

Step (iii): We claim that D is canonically isomorphic to a semidirect product $M \rtimes D/M$: Since D is a factor of S_1, we know that D is also in the variety $\mathfrak{A}_p\mathfrak{V}$. Then D is an extension of a finite group A in $\mathfrak{A}_p$ by a finite group V in $\mathfrak{V}$. But $(p, e) = 1$, hence $(|A|, |V|) = 1$ and Schur's Theorem (Theorem 2.1.7) applies to give $D = A \rtimes B$ for some subgroup $B \subseteq D$. Since A is normal in D and a vector space over $\mathbb{Z}_p$, we can define

$$\alpha : B \longrightarrow GL(A) \ (b \mapsto \bar{b}),$$

where $\bar{b}$ is defined as follows

$$\bar{b} : A \longrightarrow A \ (a \mapsto b^{-1}ab).$$

It is easy to see that α is a homomorphism and hence a linear representation of B over $\mathbb{Z}_p$. As $B \cong D/A \cong V$, we have $(|B|, p) = (|V|, p) = 1$, and, applying Maschke's Theorem (see Theorem 2.1.9), we can decompose A into a direct sum of irreducible subrepresentations. That is, $A = I_1 \oplus I_2 \oplus \ldots \oplus I_k$, where each I_i is an irreducible subrepresentation. We then obtain that $b^{-1}I_ib \subseteq I_i$ for all $b \in B$. Moreover, we know that $a^{-1}I_ia \subseteq I_i$ for all $a \in A$, since $I_i \subseteq A$ and A is abelian. This shows that I_i is normal in D for all $i = 1, \ldots, k$, and hence contains M. But all I_i's are disjoint. This forces $A = I_1$. Now we have that M is a subrepresentation of I_1 but I_1 is irreducible. This gives us that M and I_1 must be equal. Hence, we get $D = M \rtimes B$, or equivalently $D \cong M \rtimes D/M$, where the multiplication is canonically given for $m_1, m_2 \in M$, $d_1, d_2 \in D$ by

$$(m_1, d_1M)(m_2, d_2M) = (m_1 m_2^{d_1^{-1}M}, d_1d_2M).$$

Observe here that $M = A \in \mathfrak{A}_p$ is abelian, thus $M \subseteq C_D(M)$, and m^{dM} is well-defined with $m^{dM} = m^{dC_D(M)}$ for all $m \in M, d \in D$.

Step (iv): We claim that S_0 is isomorphic to a factor of D. We begin by defining the following map

$$\Theta : M \rtimes D/M \longrightarrow M \rtimes D/C_D(M) \; ((m, dM) \mapsto (m, dC_D(M))).$$

Since $M \subseteq C_D(M)$, it is easy to see that this map is well-defined and onto. If we can show that Θ is a homomorphism, then $M \rtimes D/C_D(M)$ is an epimorphic image and factor of $M \rtimes D/M$. But we have for all $m_1, m_2 \in M$, $d_1, d_2 \in D$:

$$\begin{aligned}((m_1, d_1M)(m_2, d_2M))\Theta &= (m_1 m_2^{d_1^{-1}M}, d_1d_2C_D(M)) \\ &= (m_1 m_2^{d_1^{-1}C_D(M)}, d_1d_2C_D(M)) \\ &= (m_1, d_1C_D(M))(m_2, d_2C_D(M)) \\ &= (m_1, d_1M)\Theta(m_2, d_2M)\Theta.\end{aligned}$$

From Steps (i) and (ii), we have that S_0 is isomorphic to $M \rtimes D/C_D(M)$. Thus, from Step (iii) and the epimorphism Θ, S_0 is isomorphic to a factor of D, which is a factor of S_1. Hence, we finally have that S_0 is isomorphic to a factor of S_1. Thus, there exists some epimorphism $\gamma : U \longrightarrow S_0 = M_0 \rtimes T_0$ for some subgroup $U \subseteq S_1 = M_1 \rtimes T_1$, which induces an epimorphism $\gamma' : U \longrightarrow S_0/M_0 \cong T_0$. We have $U \cap M_1 \subseteq \operatorname{Ker}\gamma'$ as the orders of the groups M_1 and T_0 are relatively prime, and γ' induces an epimorphism from $U/U \cap M_1 \cong UM_1/M_1 \subseteq S_1/M_1 \cong T_1$ onto T_0. This forces T_0 to be isomorphic to a factor of T_1, which contradicts Condition c). This completes the proof.

□

The main implication from [29] is the following

Corollary 6.1.5. *(i) For any $e = 8p_1$, where p_1 is an odd prime, the conditions of Lemma 6.1.4 are satisfied by a suitable series T_i, $i \in \omega$, of finite solvable groups of length* 4.

(ii) For any distinct odd primes p_1, p_2 there exist $2^{\aleph_0}$ pairwise distinct subvarieties of the locally finite variety of length 5 *solvable groups of exponent* $8p_1p_2$.

We apply these results to obtain the existence of $2^{\aleph_0}$ pairwise distinct varieties of groups which are not closed under cellular covers. Moreover, this answers a problem raised in [13].

Theorem 6.1.6. *There exist $2^{\aleph_0}$ pairwise distinct varieties of groups which are not cellular closed.*

Proof. Let $p_3 > 10^{75}$ be a prime distinct from p_1 and p_2, and let $\mathfrak{B}_{p_3}$ be the associated Burnside variety and $\mathcal{B} \in \mathfrak{B}_{p_3}$ the special Burnside group from Section 3.3. We have that $\mathfrak{V}\mathfrak{B}_{p_3} \neq \mathfrak{V}'\mathfrak{B}_{p_3}$ for any varieties $\mathfrak{V} \neq \mathfrak{V}'$, by applying Theorem 3.2.7. Thus, there exist $2^{\aleph_0}$ distinct product varieties

$\mathfrak{W}\mathfrak{B}_{p_3}$, where $\mathfrak{W}$ is a variety obtained from Corollary 6.1.5 (ii). It is clear that $\mathfrak{W}\mathfrak{B}_{p_3}$ contains the group $\mathcal{B}$, as $\mathcal{B} \in \mathfrak{B}_{p_3} \subseteq \mathfrak{W}\mathfrak{B}_{p_3}$. Since $\mathfrak{W}$ is of exponent $8p_1p_2$ and $\mathfrak{B}_{p_3}$ is of exponent p_3, the product $\mathfrak{W}\mathfrak{B}_{p_3}$ is of exponent $8p_1p_2p_3$. But, as shown in Section 5.4, there exists a cellular cover $e : A \longrightarrow \mathcal{B}$ with $\mathbb{Z} \subseteq A$, i.e., A is not in the product variety $\mathfrak{W}\mathfrak{B}_{p_3}$. This shows the existence of $2^{\aleph_0}$ pairwise distinct varieties which are not closed under cellular covers. □

This result actually strengthens Theorem 5.4.6.

Corollary 6.1.7. *There exist $2^{\aleph_0}$ pairwise distinct varieties of groups which are neither cellular closed nor finitely based.*

Proof. From Theorem 6.1.6 we obtain $2^{\aleph_0}$ varieties which are not cellular closed. As there are only countably many finitely based varieties, there actually must exist $2^{\aleph_0}$ pairwise distinct varieties which are neither cellular closed nor finitely based. □

Remark 6.1.8. *Observe that the existence of $2^{\aleph_0}$ pairwise distinct not finitely based varieties in Corollary 6.1.7 is derived only implicitly by set-theoretic considerations, while in Theorem 5.4.6 countably many such varieties are explicitly given.*

6.2 Embeddings into the lattice of all varieties of groups

By G. Birkhoff's theorem any (group) variety is characterized as a closed class of groups, which means that it is closed under taking cartesian products, quotients and subgroups (see [27, pp. 15–17] for more details). Hence, it is easy to see that the varieties constitute a lattice $\mathcal{G}$ under set inclusion, where the infimum of two varieties is their intersection, and the supremum is the variety obtained as the closure of their union. We want to investigate to what extent the lattice $\mathfrak{P}(\omega)$ of subsets of ω can be embedded into $\mathcal{G}$, using the existence of varieties which are not finitely based.

In Lemma 6.1.4, we did check (6.1) for the varieties $\mathfrak{U}_I := \mathrm{var}\{S_i \mid i \in I\}$, $I \subseteq \omega$, where $S_i = M_i \rtimes T_i$, $i \in \omega$. Thus, we can conclude the following

Theorem 6.2.1. *The mapping $\alpha : \mathcal{P}(\omega) \longrightarrow \mathcal{G}$ $(I \mapsto \mathfrak{U}_I = \mathrm{var}\{S_i \mid i \in I\})$ is an order embedding, where $\alpha(\emptyset)$ is the trivial variety $\{1\}$ and $\alpha(\omega)$ is the variety generated by all groups S_i, $i \in \omega$.*

Proof. In the proof of (6.1) we did show that $S_i \notin \mathrm{var}\{S_j \mid i \neq j \in \omega\}$. Thus, for $I, J \subseteq \omega$ holds $I \subseteq J$ if and only if $\alpha(I) \subseteq \alpha(J)$. □

Note, however, that the order embedding α fails to preserve the lattice structure of $\mathfrak{P}(\omega)$: Let $I := \{0\}$ and $J := \{1\}$. Thus, $I \cap J = \emptyset$ and $\alpha(I \cap J) = \{1\}$, while $\mathbb{Z}_p \subseteq M_0 \subseteq S_0 \in \alpha(I)$, $\mathbb{Z}_p \subseteq M_1 \subseteq S_1 \in \alpha(J)$ and $\mathbb{Z}_p \in \alpha(I) \cap \alpha(J)$. Hence, $\alpha(I \cap J) \neq \alpha(I) \cap \alpha(J)$, and the following lemma shows that this is a problem of a general nature.

Lemma 6.2.2. *There exists no lattice embedding $\alpha : \mathfrak{P}(\omega) \longrightarrow \mathcal{G}$.*

Proof. Let us assume the existence of a lattice embedding $\alpha : \mathfrak{P}(\omega) \longrightarrow \mathcal{G}$. We set $\mathfrak{D}_i := \alpha(\{i\})$, $i \in \omega$. Obviously, $\mathfrak{D}_i \neq \alpha(\emptyset) = \{1\}$, as α is an embedding. Thus, each $\mathfrak{D}_i$ does contain non-trivial groups and non-trivial cyclic groups as subgroups. Furthermore, $\mathbb{Z} \in \mathfrak{D}_i$ implies $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z} \in \mathfrak{D}_i$, $n > 1$, as $\mathfrak{D}_\mathfrak{i}$ is closed under quotients. Thus, for each $i \in \omega$ we may choose some $n_i > 1$ with $\mathbb{Z}_{n_i} \in \mathfrak{D}_\mathfrak{i}$.

We claim that $i \neq j$ implies $n_i \neq n_j$: Let be $i \neq j$ with $n_i = n_j$. Then

$$\mathfrak{D}_\mathfrak{i} \cap \mathfrak{D}_\mathfrak{j} = \alpha(\{i\}) \cap \alpha(\{j\}) = \alpha(\{i\} \cap \{j\}) = \alpha(\emptyset) = \{1\},$$

but $1 \neq \mathbb{Z}_{n_i} \in \mathfrak{D}_\mathfrak{i} \cap \mathfrak{D}_\mathfrak{j}$, a contradiction. Thus, the set $\{n_{2k} \mid k \in \omega\}$ is unbounded, and the element $(1 \mid k \in \omega) \in \prod_{k\in\omega} \mathbb{Z}_{n_{2k}} \in \alpha(\{2k \mid k \in \omega\})$ has infinite order. We conclude $\mathbb{Z} \in \alpha(\{2k \mid k \in \omega\})$, and, quite similarly, $\mathbb{Z} \in \alpha(\{2k+1 \mid k \in \omega\})$ holds. Hence,

$$\begin{aligned}
\mathbb{Z} &\in \alpha(\{2k \mid k \in \omega\}) \cap \alpha(\{2k+1 \mid k \in \omega\}) \\
&= \alpha(\{2k \mid k \in \omega\} \cap \{2k+1 \mid k \in \omega\}) \\
&= \alpha(\emptyset) \\
&= \{1\},
\end{aligned}$$

which is a contradiction and completes the proof. □

In set theory, it is known that the structure of the power set $\mathcal{P}(\omega)$ of ω is complicated. To illustrate this, we will consider chains and antichains in $\mathcal{P}(\omega)$. We first recall the following definitions from [20].

Definition 6.2.3. *Let $B \subseteq A$, where $(A, \leq)$ is a poset. We say that B is a* chain *in A if any two elements of B are comparable, i.e., if $a, b \in B$, then either $a \leq b$ or $b \leq a$, and we say that B is an* antichain *in A if any two distinct elements of B are incomparable, i.e., if $a, b \in B$ and $a \neq b$, then neither $a \leq b$ nor $b \leq a$.*

Obviously, many chains in $\mathcal{P}(\omega)$ have cardinality $\aleph_0$. However, there are also chains of larger size. Here is an example of an uncountable chain: For each

$r \in \mathbb{R}$, set $A_r := \{q \in \mathbb{Q} \mid q \leq r\}$. We can see that the set $C := \{A_r \mid r \in \mathbb{R}\}$ is a chain in $\mathcal{P}(\mathbb{Q})$, hence a chain in $\mathcal{P}(\omega)$, since $\mathbb{Q}$ is countable. But $|C| = |\mathbb{R}| = 2^{\aleph_0}$, which is the cardinality of $\mathcal{P}(\omega)$. In addition, we will consider two important types of antichains in $\mathcal{P}(\omega)$, namely, disjoint families and almost disjoint families. A family of sets is *disjoint* if any two of its members are disjoint, see [21, p. 12]. We see that the cardinality of any disjoint family in $\mathcal{P}(\omega)$ cannot exceed $\aleph_0$, since there exist only $\aleph_0$ many elements to make members of the family disjoint. But there exist antichains of cardinality $2^{\aleph_0}$ as can be shown via a suitable almost disjoint family. We elaborate this by starting with the following

Definition 6.2.4. *If X and Y are infinite subsets of ω, then X and Y are* almost disjoint *if $X \cap Y$ is finite.*

We refer to [21, p. 118] for the previous definition and the next

Lemma 6.2.5. *There exists an almost disjoint family of $2^{\aleph_0}$ subsets of ω.*

Proof. For any ordinal α let ${}^{\alpha}2 := \{f \mid f : \alpha \longrightarrow \{0,1\}\}$. Note that the set of all functions from a finite ordinal to $\{0,1\}$, denoted by ${}^{\omega>}2$, is countably infinite and may be identified with ω. Moreover, the set ${}^{\omega}2$ has the cardinality $2^{\aleph_0}$.

For every $f : \omega \longrightarrow \{0,1\}$, let A_f be the set

$$A_f := \{f \restriction n \mid n \in \omega\} \subseteq {}^{\omega>}2.$$

It is clear that $|A_f| = \aleph_0$, and $A_f \cap A_g$ is finite if $f \neq g$. Therefore, the family $F := \{A_f \mid f \in {}^{\omega}2\}$ consists of $2^{\aleph_0}$ almost disjoint subsets of ${}^{\omega>}2$ (and ω). $\square$

The structure of the power set $\mathcal{P}(\omega)$ is surprisingly difficult and exhibits a number of quite remarkable undecidability problems: It is, for instance, undecidable from standard set theory ZFC, whether the continuum hypothesis $2^{\aleph_0} = \aleph_1$ (CH) holds, i.e., whether the cardinality of $\mathcal{P}(\omega)$ is equal to the smallest uncountable cardinal. This means that in some models of set theory, most notably in Gödel's constructible universe V=L, CH is true, while there exist other consistent models of ZFC in which CH fails. Quite similarly, it is undecidable, whether every maximal almost disjoint family in $\mathcal{P}(\omega)$ has cardinality $2^{\aleph_0}$. However, this statement holds true provided that CH holds, see [23, p. 48].

As we can order embed $(\mathcal{P}(\omega), \subseteq)$ into $(\mathcal{G}, \subseteq)$, it is not surprising that $\mathcal{G}$ should show a similarly difficult order structure, and as an immediate consequence of Theorem 6.2.1 and Corollary 6.1.5, we get the following

Corollary 6.2.6. *The poset $(\mathcal{P}(\omega), \subseteq)$ order embeds into the poset of subvarieties of the Burnside variety $\mathfrak{B}_{8p_1p_2}$ for any distinct odd primes p_1 and p_2.*

Chapter 7

Localizations of groups: the dual case

Cellular covers are also known as co-localizations. Hence, the study of cellular covers can be viewed as dual to the study of localizations, and in this chapter we will review a result dual to Theorem 6.1.6 for localizations from our joint paper with R. Göbel and J. L. Rodríguez [14].

7.1 $2^{\aleph_0}$ varieties of groups not closed under localizations

We can see the resemblance between cellular covers and localizations in the following definition which can be found in [15] and [16].

Definition 7.1.1. *A group homomorphism $e : G \longrightarrow H$ is a* localization *of G if any homomorphism $\varphi \in \mathrm{Hom}(G, H)$ factors uniquely through e, i.e., there is a unique homomorphism $\overline{\varphi} : H \longrightarrow H$ such that $e\overline{\varphi} = \varphi$.*

It is equivalent to saying that $e : G \longrightarrow H$ is a localization of G if the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "e"] \arrow[dr, "\forall\varphi"'] & H \arrow[d, dotted, "\exists!\overline{\varphi}"] \\ & H \end{tikzcd}$$

Like with cellular covers, it was investigated which properties of groups transfer to their localizations. Here Libman showed in [25] that localizations fail to preserve finiteness of groups, i.e., a localization of a finite group need not be finite. In particular, he proved for all even integers $n \geq 10$ that the canonical irreducible representation $\rho : A_n \longrightarrow SO(n-1, \mathbb{R}) \subseteq GL(n-1, \mathbb{R})$ is a localization of A_n, where A_n is the alternating group on n letters and $SO(n, \mathbb{R})$ is the special orthogonal group of orthogonal $n \times n$ matrices with determinant 1 over the field of real numbers. Thus, there exists a localization of the finite group A_n which has cardinality $2^{\aleph_0}$ and Dror Farjoun raised the question about the existence of an upper bound for the cardinality of localizations of A_n. The answer to this question is given by the following result, keeping in mind that A_n is non-abelian and simple for all $n \geq 5$. A proof of this result can be found in [15] under the set-theoretic assumption ZFC+GCH and, more generally, without any set-theoretic restrictions in [16].

Corollary 7.1.2. *Any non-abelian finite simple group has localizations of arbitrarily large cardinality.*

For more details, we quote the main theorem from [16] in Theorem 7.1.5 which needs the following

Definition 7.1.3. *Let $G \neq 1$ be any group with trivial center and identify $G \subseteq \operatorname{Aut}(G)$ as inner automorphisms $\operatorname{Inn}(G)$ of G. We say that G is* suitable *if the following conditions hold:*

(i) G is finite.

(ii) If $G' \subseteq \operatorname{Aut}(G)$ and $G' \cong G$, then $G' = G$.

(iii) $\operatorname{Aut}(G)$ is complete.

Recall that a group G is said to be *complete* if its center $\mathfrak{z}G$ and outer automorphism group $\operatorname{Out}(G) := \operatorname{Aut}(G)/\operatorname{Inn}(G)$ are both trivial, see e.g. [32, p. 412].

With respect to suitable groups the following two results are of importance.

Lemma 7.1.4. *Any non-abelian finite simple group is suitable.*

Proof. Let G be a non-abelian finite simple group. Then Condition (i) holds trivially and Condition (iii) is an immediate consequence of [32, p. 414, Theorem 13.5.10].

Let now $G' \subseteq \operatorname{Aut}(G)$ with $G' \cong G$ and isomorphism $\varphi : G \longrightarrow G'$ be given. We will investigate the map $\overline{\varphi} : G \longrightarrow \operatorname{Out}(G)$ which is defined by the following concatenation of homomorphisms:

$$G \xrightarrow{\varphi} G' \longrightarrow G'G/G \subseteq \operatorname{Out}(G).$$

As G is simple, we have either $\operatorname{Ker}\overline{\varphi} = 1$ or $\operatorname{Ker}\overline{\varphi} = G$. If $\operatorname{Ker}\overline{\varphi} = 1$, then G embeds into $\operatorname{Out}(G)$ which is a solvable group according to the Schreier conjecture. Thus, also G must be solvable. If its solvable length is 1, then G must be abelian. But G is non-abelian, a contradiction. If, however, its length is bigger than 1, then G must have a non-trivial normal subgroup. But G is simple, once again a contradiction. Hence $\operatorname{Ker}\overline{\varphi} = G$, thus $G' \subseteq G$, and even $G' = G$ as G is finite. This proves Condition (ii). □

Theorem 7.1.5. *Let $\mathcal{A}$ be a family of suitable groups and μ an infinite cardinal such that $\mu^{\aleph_0} = \mu$. Then we can find a group H of cardinality $\lambda = \mu^+$ such that the following holds:*

(i) The group H is simple. Moreover, if $1 \neq g \in H$, then any element of H is a product of at most four conjugates of g.

(ii) Any $G \in \mathcal{A}$ is a subgroup of H and any two different groups in $\mathcal{A}$ have trivial intersection 1 when considered as subgroups of H. If $\mathcal{A}$ is not empty, then $H[\mathcal{A}] = H$, where the $\mathcal{A}$-socle $H[\mathcal{A}]$ is the subgroup of H generated by all copies of groups $G \in \mathcal{A}$ in H.

(iii) Any monomorphism $\varphi : G \longrightarrow H$, $G \in \mathcal{A}$, is induced by some $h \in H$, i.e., there is an element $h \in H$ such that $\varphi = h^* \restriction G$.

(iv) If $G' \subseteq H$ is an isomorphic copy of some $G \in \mathcal{A}$, then the centralizer $C_H(G')$ is trivial.

(v) Any monomorphism $H \longrightarrow H$ is an inner automorphism.

(vi) The group H contains a free subgroup of rank μ.

Note that (vi) is an extra property of the group H derived from Construction 3.4 (i) in [16]. As an immediate consequence of Lemma 7.1.4 and Theorem 7.1.5 we have the following

Proposition 7.1.6. *(i) For any even $n \geq 10$ there exists a localization of A_n with a free subgroup of countable rank.*

(ii) If G is a non-abelian finite simple group and κ an infinite cardinal, then there exists a localization of G with a free subgroup of rank κ.

Proof. (i) This is an immediate consequence of (ii). Alternatively, we may use Libman's localization $\rho : A_n \longrightarrow SO(n-1, \mathbb{R})$: We know that $SO(3, \mathbb{R})$ and hence any $SO(n, \mathbb{R})$, $n \geq 3$, contains a free subgroup of rank 2, cf. [19, pp. 469–472]. But it is well known that a free group of rank 2 has a free subgroup of countable rank, see e.g. Kurosh [24, p. 36, Vol II].

(ii) All non-abelian finite simple groups are suitable, by Lemma 7.1.4. Thus, applying Theorem 7.1.5 for the choice $\mathcal{A} = \{G\}$ and $\mu = \kappa^{\aleph_0}$, there exists a group H containing G with the following properties:

(a) The group H is simple.

(b) Any monomorphism from G to H is a restriction of an inner automorphism of H.

(c) Any monomorphism from H to H is an inner automorphism of H.

(d) The centralizer of G in H is trivial.

(e) The group H contains a free subgroup of rank $\kappa^{\aleph_0} \geq \kappa$.

It remains to show that the inclusion

$$e : G \longrightarrow H \text{ is a localization of } G,$$

i.e., we will show that for any $\varphi : G \longrightarrow H$, there exists some $\overline{\varphi} : H \longrightarrow H$ uniquely such that $e\overline{\varphi} = \varphi$.

If $\varphi = 0$, then choose $\overline{\varphi} = 0$. We obtain that $Ge\overline{\varphi} = G\overline{\varphi} = 1 = G\varphi$. Since H is simple and $1 \neq G \subseteq \operatorname{Ker} \overline{\varphi}$, the zero map is the only possibility for $\overline{\varphi}$.

If $\varphi \neq 0$, then we obtain that $\operatorname{Ker} \varphi = 1$ because $\operatorname{Ker} \varphi \trianglelefteq G$ and G is simple. Thus, φ is a monomorphism. By (b), φ is a restriction of an inner automorphism of H, say $\varphi = y^* \restriction G$ where $y \in H$ and

$$y^* : H \longrightarrow H \ (h \mapsto y^{-1}hy).$$

Choose $\overline{\varphi} = y^*$. Then follows $e\overline{\varphi} = \varphi$.

Next, assume that $\varphi' : H \longrightarrow H$ is another homomorphism such that $e\varphi' = \varphi$. Consequently, $\varphi' \neq 0$, and hence φ' is a monomorphism of H, by (a). Thus, by (c), there is $g \in H$ such that $\varphi' = g^*$. We then obtain that

$$Gg^* = G\varphi' = Ge\varphi' = G\varphi = Ge\overline{\varphi} = G\overline{\varphi} = Gy^*.$$

Hence, $g^* = y^*$ on G, and $g^{-1}ag = y^{-1}ay$ for all $a \in G$. This shows that $yg^{-1} \in C_H(G)$. Therefore, (d) implies $y = g$, hence $\overline{\varphi} = \varphi'$. This shows the uniqueness of $\overline{\varphi}$ and completes the proof. □

Proposition 7.1.6 (ii) directly implies Corollary 7.1.2. Note also that the localizations of groups in Proposition 7.1.6 contain free subgroups, hence they cannot be in any locally finite variety. Applying this result and Corollary 6.1.5 (ii), we can conclude the existence of $2^{\aleph_0}$ varieties which are not closed under localizations in the following

Theorem 7.1.7. *There exist $2^{\aleph_0}$ pairwise distinct locally finite varieties of groups which are not closed under localizations.*

Proof. Let S be any non-abelian finite simple group and set $\mathfrak{W} := \mathrm{var}\{S\}$. Thus $\mathfrak{W}$ is locally finite, cf. [27, p. 18, Theorem 15.71]. We have that $\mathfrak{V}\mathfrak{W} \neq \mathfrak{V}'\mathfrak{W}$ for any varieties $\mathfrak{V} \neq \mathfrak{V}'$, by applying Theorem 3.2.7. Thus, there exist $2^{\aleph_0}$ distinct product varieties $\mathfrak{V}\mathfrak{W}$, where $\mathfrak{V}$ is a locally finite variety obtained from Corollary 6.1.5 (ii). It is clear that $S \in \mathfrak{W} \subseteq \mathfrak{V}\mathfrak{W}$ for all these varieties. Furthermore, $\mathfrak{V}\mathfrak{W}$ is locally finite with Corollary 3.2.4.

The above argument applies to show that none of these $2^{\aleph_0}$ varieties can be closed under localizations. This completes the proof. $\square$

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